Cost analysis — work and span

15-150

Lecture 7: September 16, 2025

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Last week

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  - recursive call corresponds to inductive hypothesis

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- In week 3 we discovered (and exploited) the correspondence between programs and **asymptotic analysis**.
  - recursive calls give rise to recurrence
  - closed form solutions of recurrences for **work**

"programs as recurrences"!

Last week

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  - recursive call corresponds to inductive hypothesis
- In week 3 we discovered (and exploited) the correspondence between programs and **asymptotic analysis**.
  - recursive calls give rise to recurrence
  - closed form solutions of recurrences for **work**
- This week, we revisit work and also look at **span**
  - today: work (sequential evaluation) [illegible] (parallel evaluation)
  - Thursday: sorting

mergesort

Work analysis: list reversal

Reversing a list

```
(* rev: int list -> int list
   REQUIRES: true
   ENSURES: rev L returns the elements of L
            in reverse order.
*)
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fun rev ([] : int list) : int list = []
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```
val [] : int list = rev []
val [4,3,2,1] : int list = rev [1,2,3,4]
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fun rev ([] : int list) : int list = []
  | rev (x::xs) =


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fun rev ([] : int list) : int list = []
  | rev (x::xs) = (rev xs) @ [x]


val [] : int list = rev []
val [4,3,2,1] : int list = rev [1,2,3,4]
```

What is the **work** of reverse?

Work analysis for rev

```
fun rev ([] : int list) : int list = []
  | rev (x::xs) = (rev xs) @ [x]
```

Work: $W_{rev}(n)$ with n the list length.

Recurrence relation:

$W_{rev}(0) = c_0$

$W_{rev}(n) = c_1 + W_{rev}(n-1) + ?$

Work analysis for rev

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```

Work: $W_{rev}(n)$ with n the list length.

Recurrence relation:

$W_{rev}(0) = c_0$

$W_{rev}(n) = c_1 + W_{rev}(n-1) + W_{@}(n-1, 1)$

Recall: $W_{@}(m_1, m_2)$ is $O(m_1)$

Finding closed form:

$W_{rev}(n) \leq c_1 + W_{rev}(n-1) + c_2 \cdot (n-1)$

$W_{rev}(n) \leq c_1 + c_2 \cdot n + W_{rev}(n-1)$

Lemma: For all list values L, `length`(`rev`(L)) ≅ `length`(L).

Work analysis for rev

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fun rev ([] : int list) : int list = []
  | rev (x::xs) = (rev xs) @ [x]
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Work: $W_{rev}(n)$ with n the list length.

Finding closed form:

$$W_{rev}(n) \leq c_1 + W_{rev}(n\text{-}1) + c_2 \cdot (n\text{-}1)$$

$$W_{rev}(n) \leq c_1 + c_2 \cdot n + W_{rev}(n\text{-}1)$$

$$\leq c_1 + c_2 \cdot n + (c_1 + c_2 \cdot (n\text{-}1) + W_{rev}(n\text{-}2))$$

$$\leq c_1 + c_2 \cdot n + (c_1 + c_2 \cdot (n\text{-}1) + (c_1 + c_2 \cdot (n\text{-}2) + W_{rev}(n\text{-}3)))$$

$$\vdots$$

$$W_{rev}(n) =$$

Work analysis for rev

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Work: $W_{rev}(n)$ with n the list length.

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&\vdots \\
W_{rev}(n) &= c_0 + n \cdot c_1 + 1/2 \cdot n \cdot (n+1) \cdot c_2
\end{aligned}
$$

Sum of first n positive integers is $1/2 \cdot n \cdot (n+1)$

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&\vdots \\
W_{rev}(n) &= c_0 + n \cdot c_1 + 1/2 \cdot n \cdot (n+1) \cdot c_2
\end{aligned}
$$

Consequently: $W_{rev}(n)$ is $O(n^2)$.

Can we do better?

Reversing a list: can we do better?

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fun trev ([] : int list, acc : int list) : int list =
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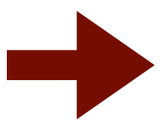
fun trev ([] : int list, acc : int list) : int list = acc
  | trev (x::xs, acc) = trev(xs, x::acc)

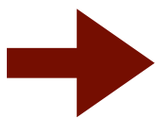
fun reverse (L : int list) : int list = trev(L, [])
```



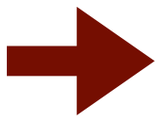
Before determining work for `trev`, are `rev` and `trev` extensional equivalent?

Are rev and trev extensional equivalent?

Theorem: For all values `L: int list` and `acc: int list`, `trev(L,acc)` ≅ `(rev L) @ acc`.



Prove this theorem as an exercise!



We provide the solution in the notes (rev.pdf). But try first!

Work analysis for trev

```
fun trev ([] : int list, acc : int list) : int list = acc
  | trev (x::xs, acc) = trev(xs, x::acc)
```

Work: $W_{trev}(n, m)$ with n the list length and m the accumulator length.

Recurrence relation:

$$W_{trev}(0, m) = c_0$$

$$W_{trev}(n, m) = c_1 + W_{trev}(n\text{-}1, m\text{+}1)$$

Finding closed form:

$$\begin{aligned} W_{trev}(n, m) &\le c_1 + W_{trev}(n\text{-}1, m\text{+}1) \\ &\le c_1 + (c_1 + W_{trev}(n\text{-}2, m\text{+}2)) \\ &\vdots \\ W_{trev}(n, m) &= c_0 + n \cdot c_1 \end{aligned}$$

Work analysis for trev

```
fun trev ([] : int list, acc : int list) : int list = acc
  | trev (x::xs, acc) = trev(xs, x::acc)
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Work: $W_{trev}(n, m)$ with n the list length and m the accumulator length.

Recurrence relation:

$W_{trev}(0, m) = c_0$

$W_{trev}(n, m) = c_1 + W_{trev}(n-1, m+1)$

Finding closed form:

$W_{trev}(n, m) = c_0 + n \cdot c_1$

Consequently: $W_{trev}(n)$ is O(n).

→ Note: Using the recurrence relation we can prove the closed form by induction on n.

Work and span analysis: tree sum

Summing integers in a tree

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```
datatype tree = Empty | Node of tree * int * tree
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fun sum (Empty : tree) : int =
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fun sum (Empty : tree) : int = 0
  |
```

Summing integers in a tree

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fun sum (Empty : tree) : int = 0
  | sum (Node(l,x,r)) =
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Summing integers in a tree

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datatype tree = Empty | Node of tree * int * tree

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   sum : tree -> int
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            all the integers in T.
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fun sum (Empty : tree) : int = 0
  | sum (Node(l,x,r)) = (sum l) + (sum r) + x
```

Work analysis for sum

```
fun sum (Empty : tree) : int = 0
  | sum (Node(l,x,r)) = (sum l) + (sum r) + x
```

Work: $W_{sum}(n)$ with n the number of integers in a tree t.

Recurrence relation:

$$W_{sum}(0) = c_0$$

$$W_{sum}(n) = c_1 + W_{sum}(n_l) + W_{sum}(n_r)$$

number of integers in left subtree

number of integers in right subtree

Work analysis for sum

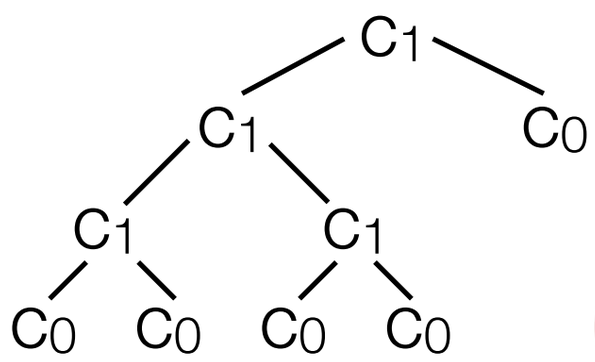
Recurrence relation:

$W_{sum}(0) = c_0$

$W_{sum}(n) = c_1 + W_{sum}(n_l) + W_{sum}(n_r)$

→ To find a closed form, let's employ the "**tree method**".

→ visualize work at each node/leaf:



A binary tree has n nodes and n+1 leaves

Closed form: $W_{sum}(n) = n \cdot c1 + (n+1) \cdot c_0$

Work analysis for sum

Recurrence relation:

$W_{sum}(0) = c_0$

$W_{sum}(n) = c_1 + W_{sum}(n_l) + W_{sum}(n_r)$

Closed form: $W_{sum}(n) = n \cdot c1 + (n+1) \cdot c_0$

Consequently: $W_{sum}(n)$ is O(n).

Is there an opportunity for parallelism?

```
fun sum (Empty : tree) : int = 0
  | sum (Node(l,x,r)) = (sum l) + (sum r) + x
```

Recurrence relation:

$W_{sum}(0) = c_0$

$W_{sum}(n) = c_1 + W_{sum}(n_l) + W_{sum}(n_r)$

number of integers in left subtree

number of integers in right subtree

→ Let's evaluate the two recursive calls in parallel!

→ Valid b/c no data dependencies between `l` and `r` (thanks to FP).

Is there an opportunity for parallelism?

```
fun sum (Empty : tree) : int = 0
  | sum (Node(l,x,r)) = (sum l) + (sum r) + x
```

Recurrence relation:

$S_{sum}(0) = c_0$

$S_{sum}(n) = c_1 + \max(S_{sum}(n_l), S_{sum}(n_r))$

number of integers in left subtree

number of integers in right subtree

→ Let's evaluate the two recursive calls in parallel!

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Span analysis for sum

```
fun sum (Empty : tree) : int = 0
  | sum (Node(l,x,r)) = (sum l) + (sum r) + x
```

Recurrence relation:

$S_{sum}(0) = c_0$

$S_{sum}(n) = c_1 + \max(S_{sum}(n_l), S_{sum}(n_r))$

Finding closed form with no balance assumption:

$S_{sum}(n) \leq c_1 + \max(S_{sum}(n\text{-}1), S_{sum}(0))$

$\leq c_1 + S_{sum}(n\text{-}1)$

Consequently: $S_{sum}(n)$ is O(n), w/o balance.

Span analysis for sum

```
fun sum (Empty : tree) : int = 0
  | sum (Node(l,x,r)) = (sum l) + (sum r) + x
```

Recurrence relation:

$S_{sum}(0) = c_0$

$S_{sum}(n) = c_1 + \max(S_{sum}(n_l), S_{sum}(n_r))$

Finding closed form with balance assumption:

$$
\begin{aligned}
S_{sum}(n) &\approx c_1 + \max(S_{sum}(n/2), S_{sum}(n/2)) \\
&= c_1 + S_{sum}(n/2) \\
&= c_1 + c_1 + S_{sum}(n/4) \\
&\vdots \\
S_{sum}(n) &= c_0 + (\lfloor \log n \rfloor) \cdot c_1
\end{aligned}
$$

Consequently: $S_{sum}(n)$ is O(log n), with balance.

Span analysis for sum (with depth)

```
fun sum (Empty : tree) : int = 0
  | sum (Node(l,x,r)) = (sum l) + (sum r) + x
```

Recurrence relation, now with depth **d** rather than number of integers:

$S_{sum}(0) = c_0$

$S_{sum}(d) = c_1 + \max(S_{sum}(d\text{-}1), S_{sum}(d'))$

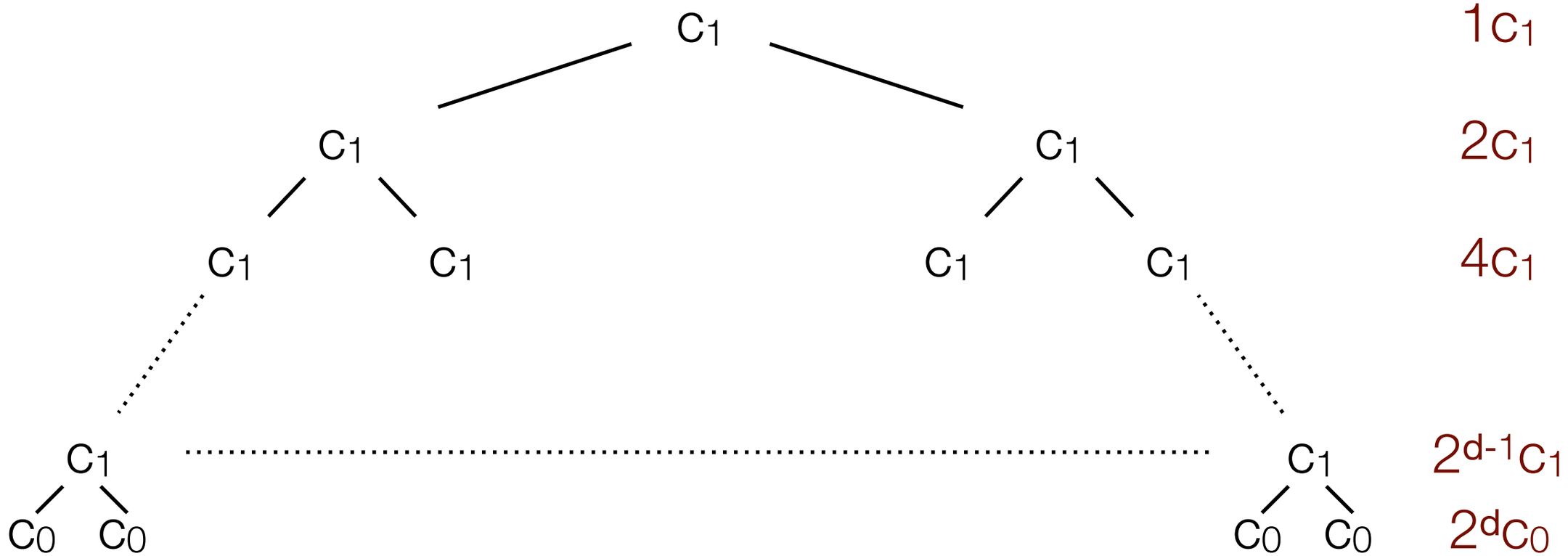
$d' \leq d\text{-}1$

Finding closed form:

$$\begin{aligned} S_{sum}(d) &\approx c_1 + S_{sum}(d\text{-}1) \\ &\vdots \\ S_{sum}(n) &= c_0 + d \cdot c_1 \end{aligned}$$

Consequently: $S_{sum}(d)$ is $O(d)$, where $d = \log(n)$, if balanced.

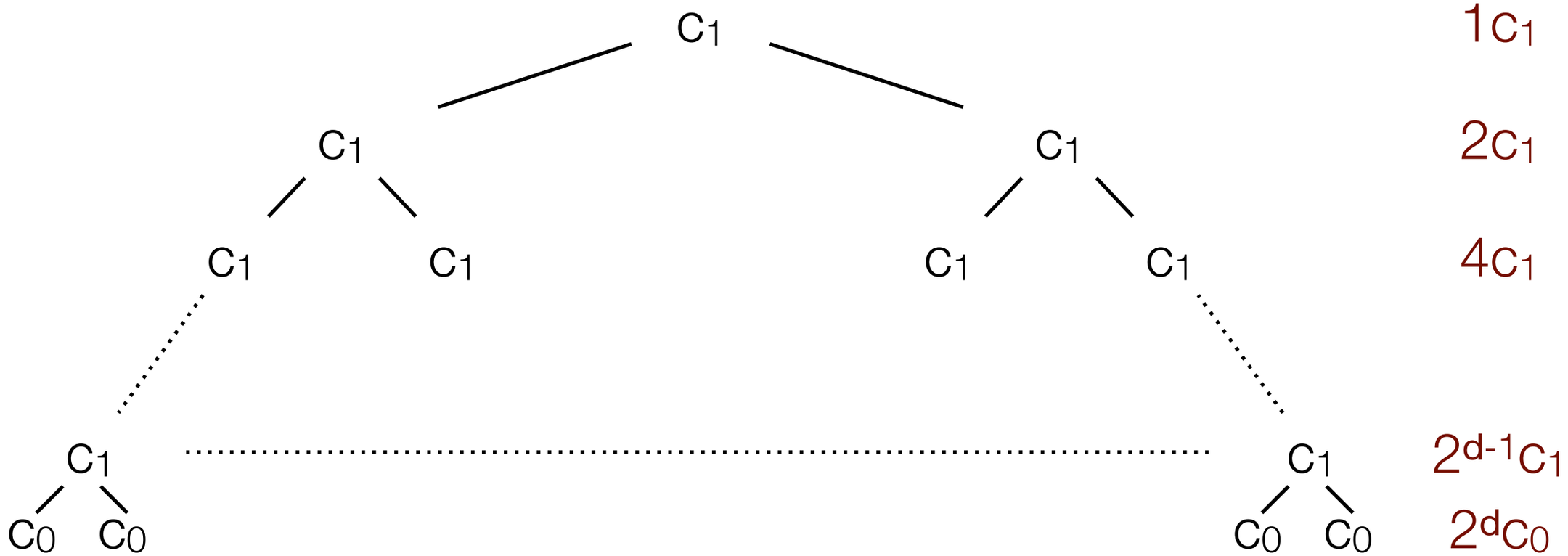
Visualize work/span for sum (with depth)



$$W_{sum}(d) = (2^0 + 2^1 + \ldots + 2^{d-1}) \cdot c_1 + 2^d \cdot c_0 \leq 2^{d+1} \cdot c$$

$< 2^d$ $\qquad$ $\max(c_0, c_1)$

Visualize work/span for sum (with depth)



$$W_{sum}(d) = (2^0 + 2^1 + \ldots + 2^{d-1})\cdot c_1 + 2^d \cdot c_0 \leq 2^{d+1}\cdot c$$

$$S_{sum}(d) = (1 + 1 + \ldots + 1)\cdot c_1 + c_0 \leq d \cdot c$$

d-1 times max(c_0, c_1)

That's all for today. See you on Thursday!