Cost analysis — work and span

15-150

Lecture 7: September 16, 2025

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Carnegie Mellon University

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mergesort

Work analysis: list reversal

Reversing a list

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(* rev: int list -> int list
   REQUIRES: true
   ENSURES: rev L returns the elements of L
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What is the **work** of reverse?

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Lemma: For all list values L, `length`(`rev`(L)) $\cong$ `length`(L).

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Sum of first n positive integers is $1/2 \cdot n \cdot (n+1)$

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Consequently: $W_{rev}(n)$ is $O(n^2)$.

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Before determining work for `trev`, are `rev` and `trev` extensional equivalent?

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- Prove this theorem as an exercise!
- We provide the solution in the notes (rev.pdf). But try first!

Work analysis for trev

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Consequently: $W_{\mathtt{trev}}(n)$ is O(n).

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Note: Using the recurrence relation we can prove the closed form by induction on n.

Work and span analysis: tree sum

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visualize work at each node/leaf:

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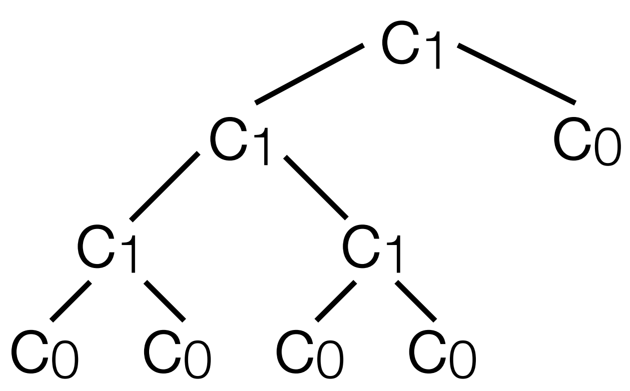

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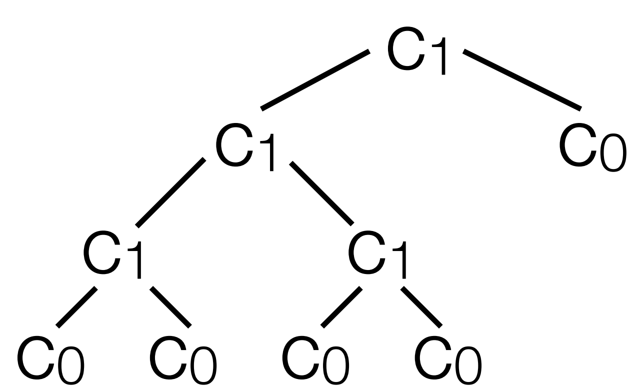

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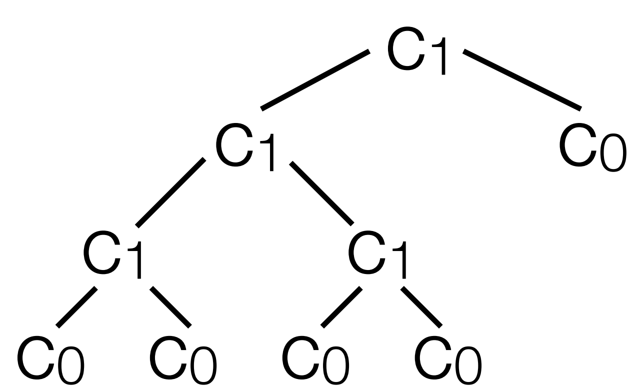

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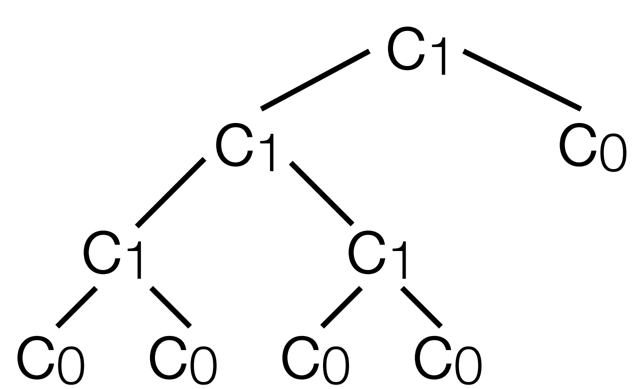

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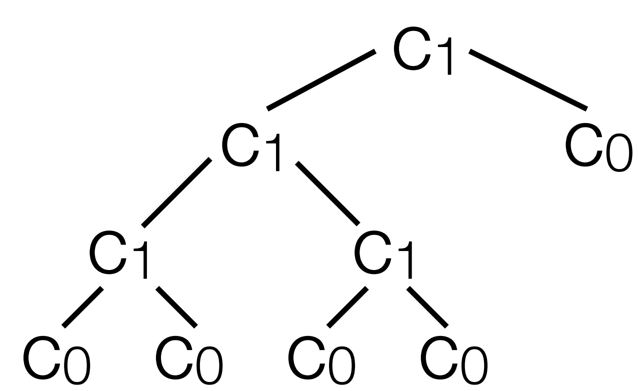

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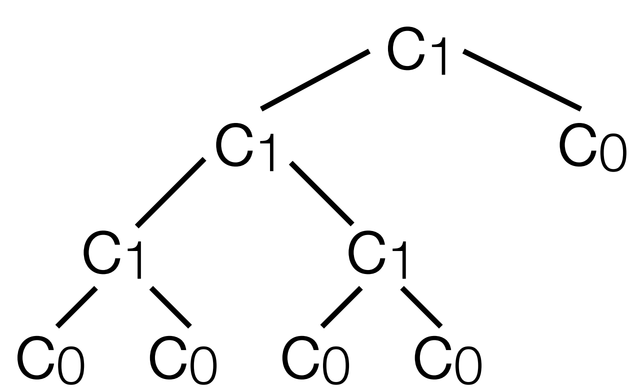

A binary tree has n nodes and n+1 leaves

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Valid b/c no data dependencies between `l` and `r` (thanks to FP).

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Recurrence relation:

$S_{sum}(0) = c_0$

$S_{sum}(n) = c_1 + \max(S_{sum}(n_l), S_{sum}(n_r))$

Finding closed form with no balance assumption:

$S_{sum}(n) \leq c_1 + \max(S_{sum}(n\text{-}1), S_{sum}(0))$

$\leq c_1 + S_{sum}(n\text{-}1)$

Span analysis for sum

```
fun sum (Empty : tree) : int = 0
  | sum (Node(l,x,r)) = (sum l) + (sum r) + x
```

Recurrence relation:

$S_{sum}(0) = c_0$

$S_{sum}(n) = c_1 + \max(S_{sum}(n_l), S_{sum}(n_r))$

Finding closed form with no balance assumption:

$$
\begin{aligned}
S_{sum}(n) &\leq c_1 + \max(S_{sum}(n\text{-}1), S_{sum}(0)) \\
&\leq c_1 + S_{sum}(n\text{-}1)
\end{aligned}
$$

Consequently: $S_{sum}(n)$ is O(n), w/o balance.

Span analysis for sum

```
fun sum (Empty : tree) : int = 0
  | sum (Node(l,x,r)) = (sum l) + (sum r) + x
```

Recurrence relation:

$S_{sum}(0) = c_0$

$S_{sum}(n) = c_1 + \max(S_{sum}(n_l), S_{sum}(n_r))$

Span analysis for sum

```
fun sum (Empty : tree) : int = 0
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Recurrence relation:

$S_{sum}(0) = c_0$

$S_{sum}(n) = c_1 + \max(S_{sum}(n_l), S_{sum}(n_r))$

Finding closed form with balance assumption:

Span analysis for sum

```
fun sum (Empty : tree) : int = 0
  | sum (Node(l,x,r)) = (sum l) + (sum r) + x
```

Recurrence relation:

$S_{sum}(0) = c_0$

$S_{sum}(n) = c_1 + \max(S_{sum}(n_l), S_{sum}(n_r))$

Finding closed form with balance assumption:

$S_{sum}(n) \approx$

Span analysis for sum

```
fun sum (Empty : tree) : int = 0
  | sum (Node(l,x,r)) = (sum l) + (sum r) + x
```

Recurrence relation:

$S_{sum}(0) = c_0$

$S_{sum}(n) = c_1 + \max(S_{sum}(n_l), S_{sum}(n_r))$

Finding closed form with balance assumption:

$S_{sum}(n) \approx c_1 + \max(S_{sum}(n/2), S_{sum}(n/2))$

Span analysis for sum

```
fun sum (Empty : tree) : int = 0
  | sum (Node(l,x,r)) = (sum l) + (sum r) + x
```

Recurrence relation:

$S_{sum}(0) = c_0$

$S_{sum}(n) = c_1 + \max(S_{sum}(n_l), S_{sum}(n_r))$

Finding closed form with balance assumption:

$$\begin{aligned} S_{sum}(n) &\approx c_1 + \max(S_{sum}(n/2), S_{sum}(n/2)) \\ &= c_1 + S_{sum}(n/2) \end{aligned}$$

Span analysis for sum

```
fun sum (Empty : tree) : int = 0
  | sum (Node(l,x,r)) = (sum l) + (sum r) + x
```

Recurrence relation:

$S_{sum}(0) = c_0$

$S_{sum}(n) = c_1 + \max(S_{sum}(n_l), S_{sum}(n_r))$

Finding closed form with balance assumption:

$$
\begin{aligned}
S_{sum}(n) &\approx c_1 + \max(S_{sum}(n/2), S_{sum}(n/2)) \\
&= c_1 + S_{sum}(n/2) \\
&= c_1 + c_1 + S_{sum}(n/4)
\end{aligned}
$$

Span analysis for sum

```
fun sum (Empty : tree) : int = 0
  | sum (Node(l,x,r)) = (sum l) + (sum r) + x
```

Recurrence relation:

$S_{sum}(0) = c_0$

$S_{sum}(n) = c_1 + \max(S_{sum}(n_l), S_{sum}(n_r))$

Finding closed form with balance assumption:

$$
\begin{aligned}
S_{sum}(n) &\approx c_1 + \max(S_{sum}(n/2), S_{sum}(n/2)) \\
&= c_1 + S_{sum}(n/2) \\
&= c_1 + c_1 + S_{sum}(n/4) \\
&\vdots
\end{aligned}
$$

Span analysis for sum

```
fun sum (Empty : tree) : int = 0
  | sum (Node(l,x,r)) = (sum l) + (sum r) + x
```

Recurrence relation:

$S_{sum}(0) = c_0$

$S_{sum}(n) = c_1 + \max(S_{sum}(n_l), S_{sum}(n_r))$

Finding closed form with balance assumption:

$$
\begin{aligned}
S_{sum}(n) &\approx c_1 + \max(S_{sum}(n/2), S_{sum}(n/2)) \\
&= c_1 + S_{sum}(n/2) \\
&= c_1 + c_1 + S_{sum}(n/4) \\
&\vdots \\
S_{sum}(n) &= c_0 + (\lfloor \log n \rfloor) \cdot c_1
\end{aligned}
$$

Span analysis for sum

```
fun sum (Empty : tree) : int = 0
  | sum (Node(l,x,r)) = (sum l) + (sum r) + x
```

Recurrence relation:

$S_{sum}(0) = c_0$

$S_{sum}(n) = c_1 + \max(S_{sum}(n_l), S_{sum}(n_r))$

Finding closed form with balance assumption:

$$
\begin{aligned}
S_{sum}(n) &\approx c_1 + \max(S_{sum}(n/2), S_{sum}(n/2)) \\
&= c_1 + S_{sum}(n/2) \\
&= c_1 + c_1 + S_{sum}(n/4) \\
&\vdots \\
S_{sum}(n) &= c_0 + (\lfloor \log n \rfloor) \cdot c_1
\end{aligned}
$$

Consequently: $S_{sum}(n)$ is O(log n), with balance.

Span analysis for sum (with depth)

```
fun sum (Empty : tree) : int = 0
  | sum (Node(l,x,r)) = (sum l) + (sum r) + x
```

Span analysis for sum (with depth)

```
fun sum (Empty : tree) : int = 0
  | sum (Node(l,x,r)) = (sum l) + (sum r) + x
```

Recurrence relation, now with depth **d** rather than number of integers:

Span analysis for sum (with depth)

```
fun sum (Empty : tree) : int = 0
  | sum (Node(l,x,r)) = (sum l) + (sum r) + x
```

Recurrence relation, now with depth **d** rather than number of integers:

$S_{sum}(0) = c_0$

Span analysis for sum (with depth)

```
fun sum (Empty : tree) : int = 0
  | sum (Node(l,x,r)) = (sum l) + (sum r) + x
```

Recurrence relation, now with depth **d** rather than number of integers:

$S_{sum}(0) = c_0$

$S_{sum}(d) =$

Span analysis for sum (with depth)

```
fun sum (Empty : tree) : int = 0
  | sum (Node(l,x,r)) = (sum l) + (sum r) + x
```

Recurrence relation, now with depth **d** rather than number of integers:

$S_{sum}(0) = c_0$

$S_{sum}(d) = c_1 + \max(S_{sum}(d-1), S_{sum}(d'))$

Span analysis for sum (with depth)

```
fun sum (Empty : tree) : int = 0
  | sum (Node(l,x,r)) = (sum l) + (sum r) + x
```

Recurrence relation, now with depth **d** rather than number of integers:

$S_{sum}(0) = c_0$

$S_{sum}(d) = c_1 + \max(S_{sum}(d-1), S_{sum}(d'))$

d' ≤ d-1

Span analysis for sum (with depth)

```
fun sum (Empty : tree) : int = 0
  | sum (Node(l,x,r)) = (sum l) + (sum r) + x
```

Recurrence relation, now with depth **d** rather than number of integers:

$S_{sum}(0) = c_0$

$S_{sum}(d) = c_1 + \max(S_{sum}(d-1), S_{sum}(d'))$

d' ≤ d-1

Finding closed form:

Span analysis for sum (with depth)

```
fun sum (Empty : tree) : int = 0
  | sum (Node(l,x,r)) = (sum l) + (sum r) + x
```

Recurrence relation, now with depth **d** rather than number of integers:

$S_{sum}(0) = c_0$

$S_{sum}(d) = c_1 + \max(S_{sum}(d\text{-}1), S_{sum}(d'))$

d' ≤ d-1

Finding closed form:

$S_{sum}(d) \approx$

Span analysis for sum (with depth)

```
fun sum (Empty : tree) : int = 0
  | sum (Node(l,x,r)) = (sum l) + (sum r) + x
```

Recurrence relation, now with depth **d** rather than number of integers:

$S_{sum}(0) = c_0$

$S_{sum}(d) = c_1 + \max(S_{sum}(d-1), S_{sum}(d'))$

$d' \leq d-1$

Finding closed form:

$S_{sum}(d) \approx c_1 + S_{sum}(d-1)$

Span analysis for sum (with depth)

```
fun sum (Empty : tree) : int = 0
  | sum (Node(l,x,r)) = (sum l) + (sum r) + x
```

Recurrence relation, now with depth **d** rather than number of integers:

$S_{\mathtt{sum}}(0) = c_0$

$S_{\mathtt{sum}}(d) = c_1 + \max(S_{\mathtt{sum}}(d\text{-}1), S_{\mathtt{sum}}(d'))$

$d' \leq d\text{-}1$

Finding closed form:

$S_{\mathtt{sum}}(d) \approx c_1 + S_{\mathtt{sum}}(d\text{-}1)$

$\vdots$

Span analysis for sum (with depth)

```
fun sum (Empty : tree) : int = 0
  | sum (Node(l,x,r)) = (sum l) + (sum r) + x
```

Recurrence relation, now with depth **d** rather than number of integers:

$S_{sum}(0) = c_0$

$S_{sum}(d) = c_1 + \max(S_{sum}(d-1), S_{sum}(d'))$

d' ≤ d-1

Finding closed form:

$$
\begin{aligned}
S_{sum}(d) &\approx c_1 + S_{sum}(d-1) \\
&\vdots \\
S_{sum}(n) &= c_0 + d \cdot c_1
\end{aligned}
$$

Span analysis for sum (with depth)

```
fun sum (Empty : tree) : int = 0
  | sum (Node(l,x,r)) = (sum l) + (sum r) + x
```

Recurrence relation, now with depth **d** rather than number of integers:

$S_{sum}(0) = c_0$

$S_{sum}(d) = c_1 + \max(S_{sum}(d\text{-}1), S_{sum}(d'))$

d' ≤ d-1

Finding closed form:

$S_{sum}(d) \approx c_1 + S_{sum}(d\text{-}1)$

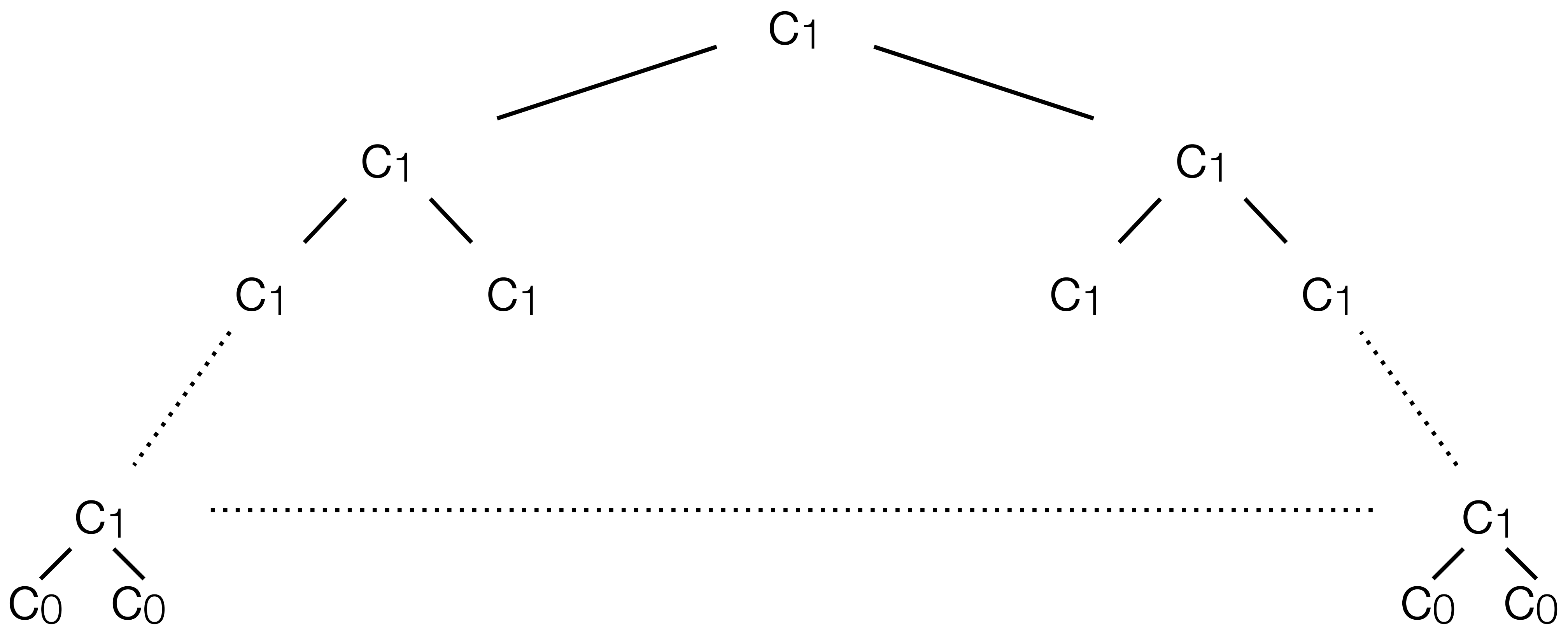
$\vdots$

$S_{sum}(n) = c_0 + d \cdot c_1$

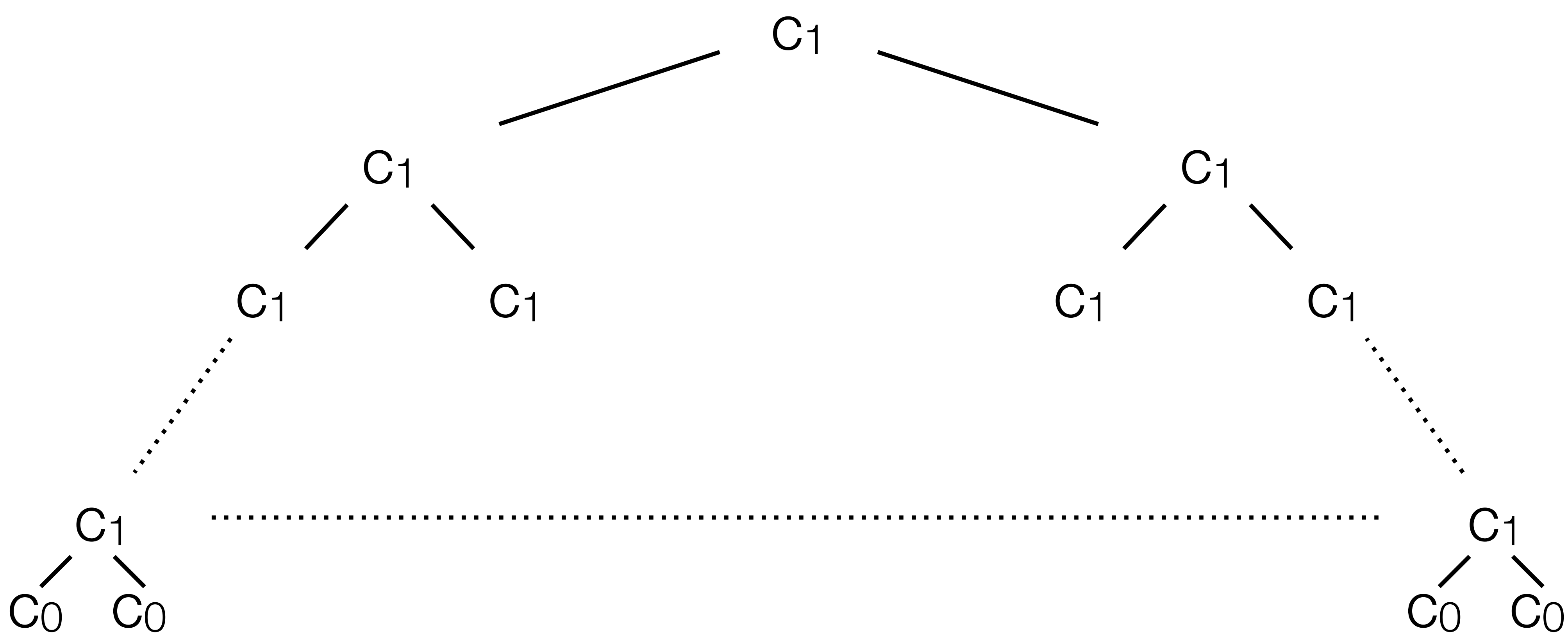
Consequently: $S_{sum}(d)$ is O(**d**), where **d** = log(**n**), if balanced.

Visualize work/span for sum (with depth)

Visualize work/span for sum (with depth)

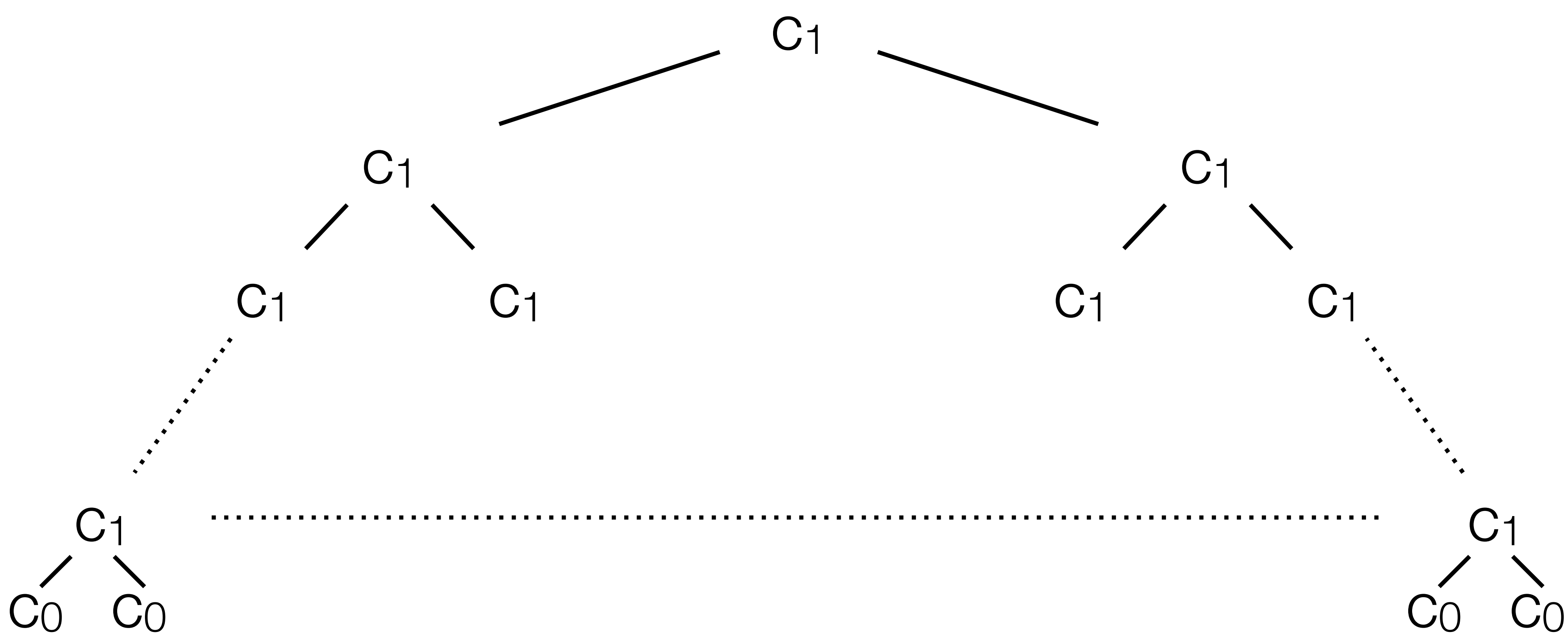


Visualize work/span for sum (with depth)



$1c_1$

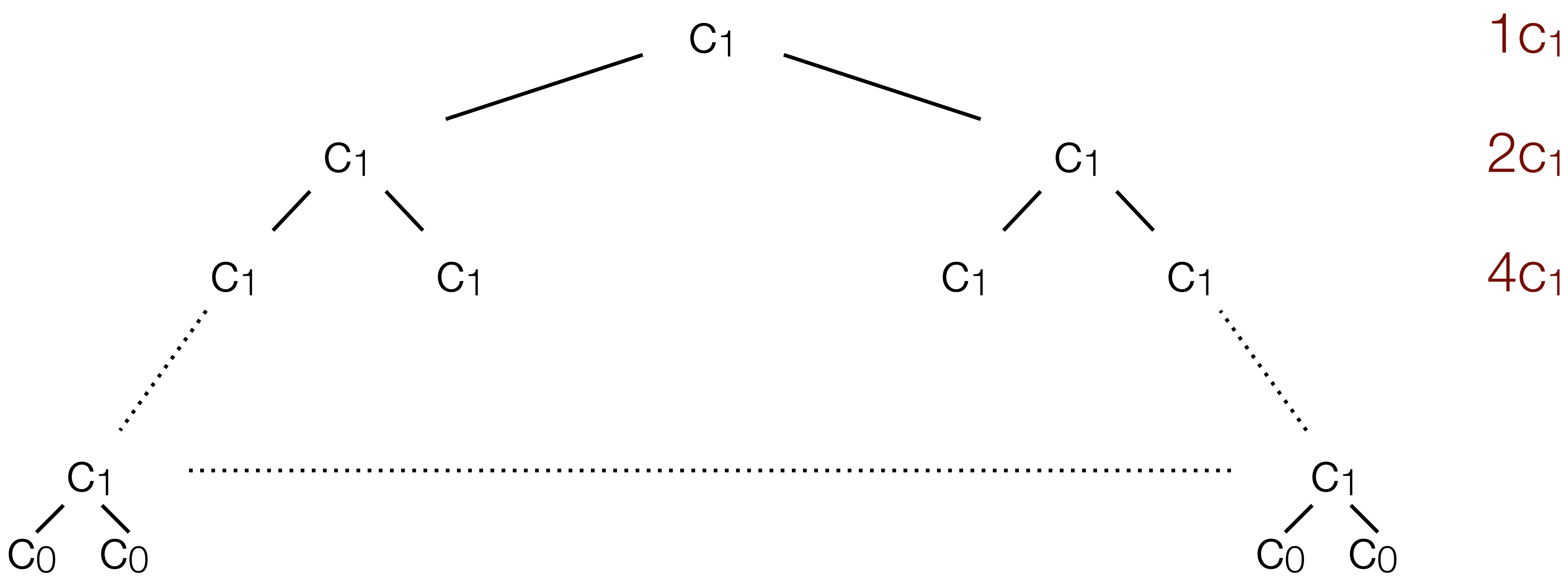
Visualize work/span for sum (with depth)



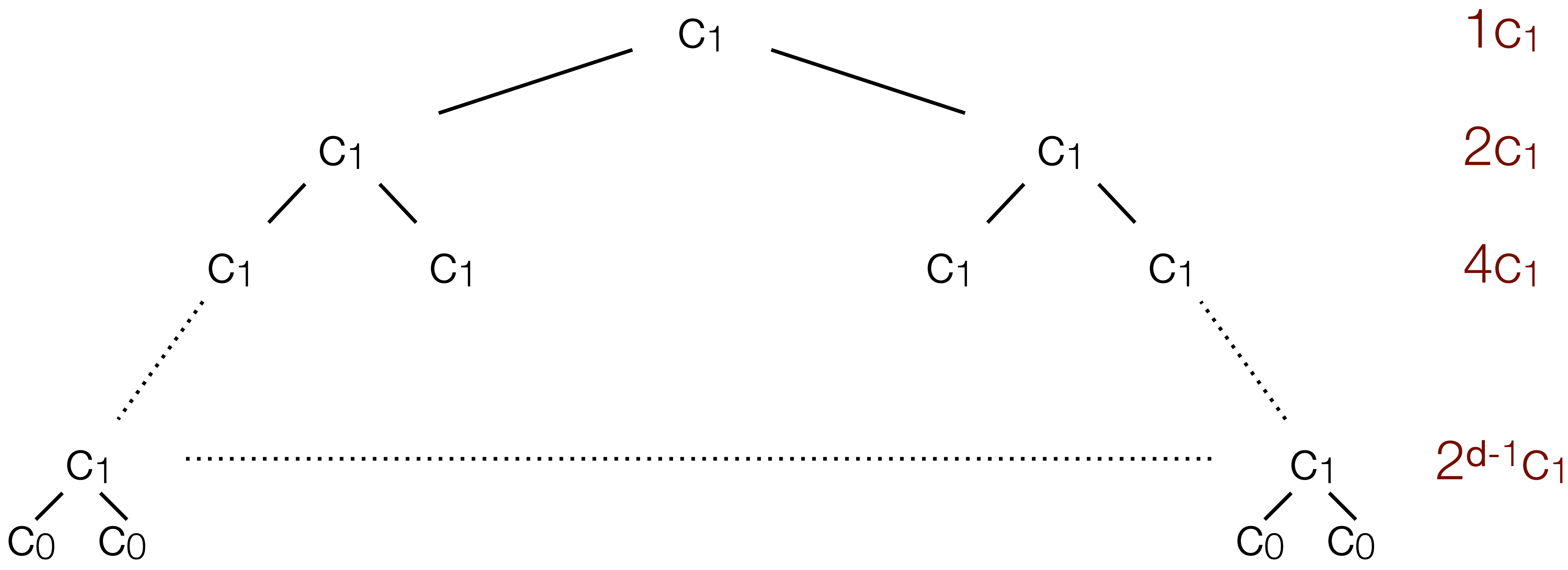
$1c_1$

$2c_1$

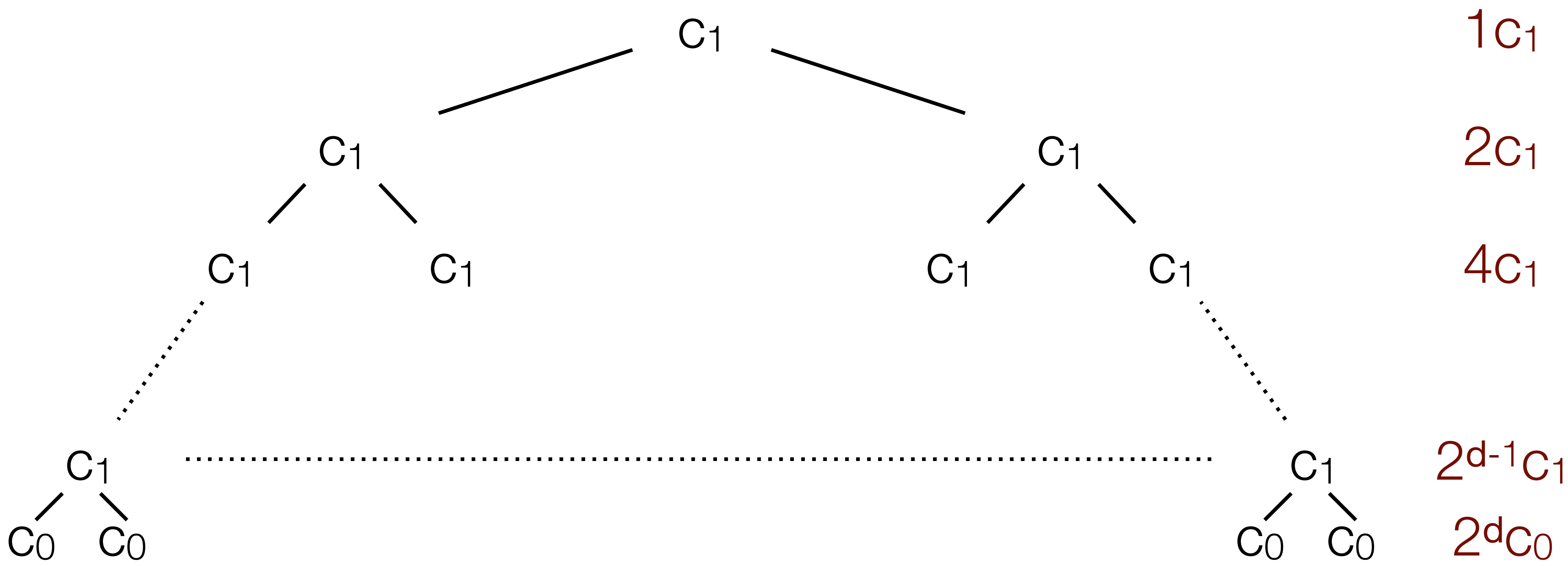
Visualize work/span for sum (with depth)



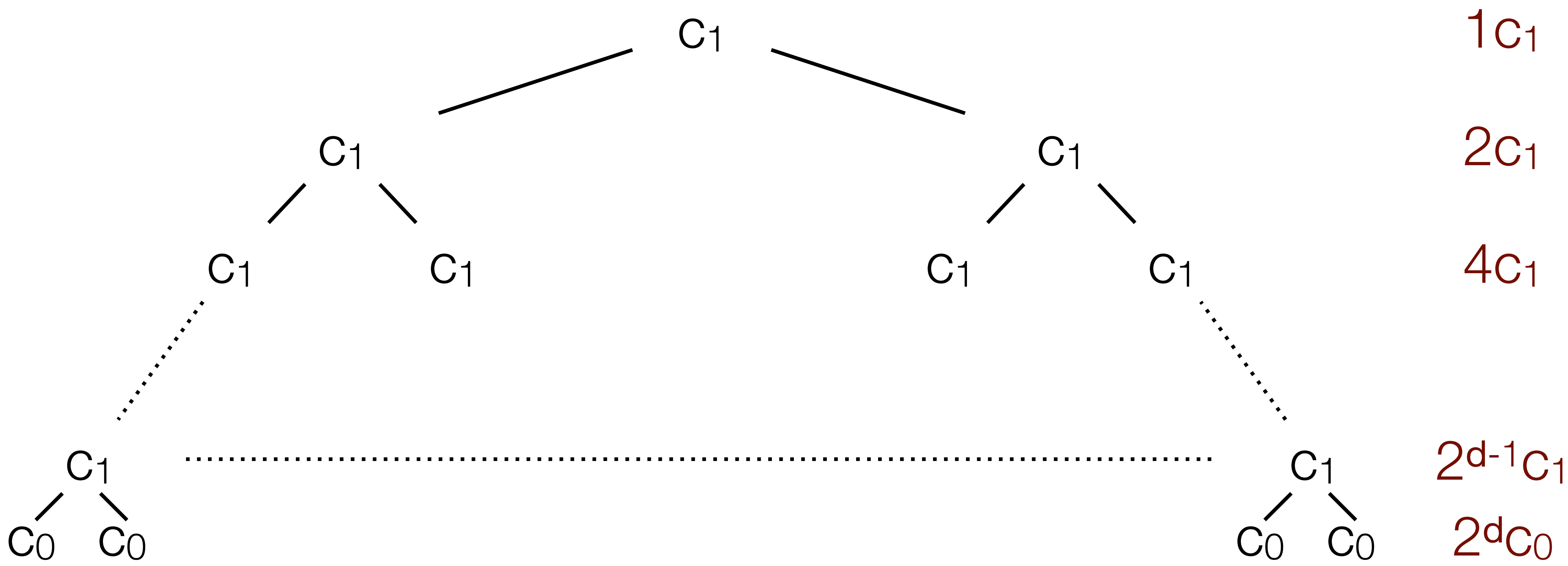
Visualize work/span for sum (with depth)



Visualize work/span for sum (with depth)

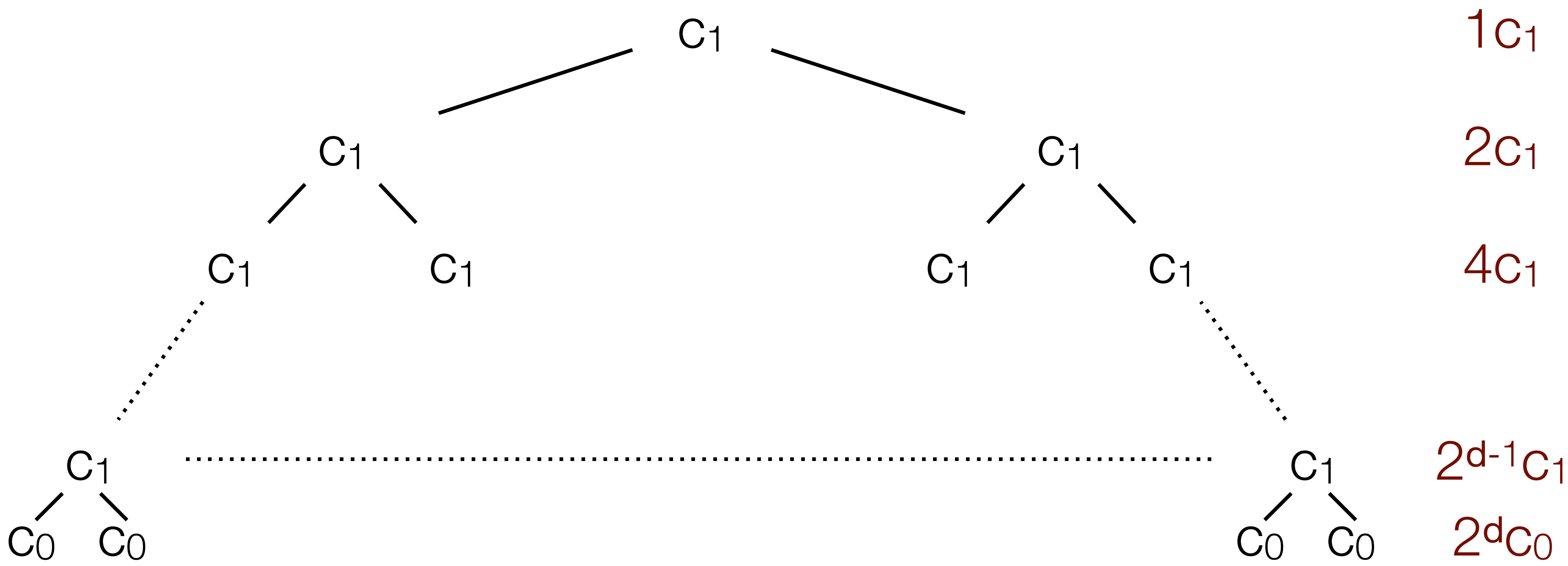


Visualize work/span for sum (with depth)



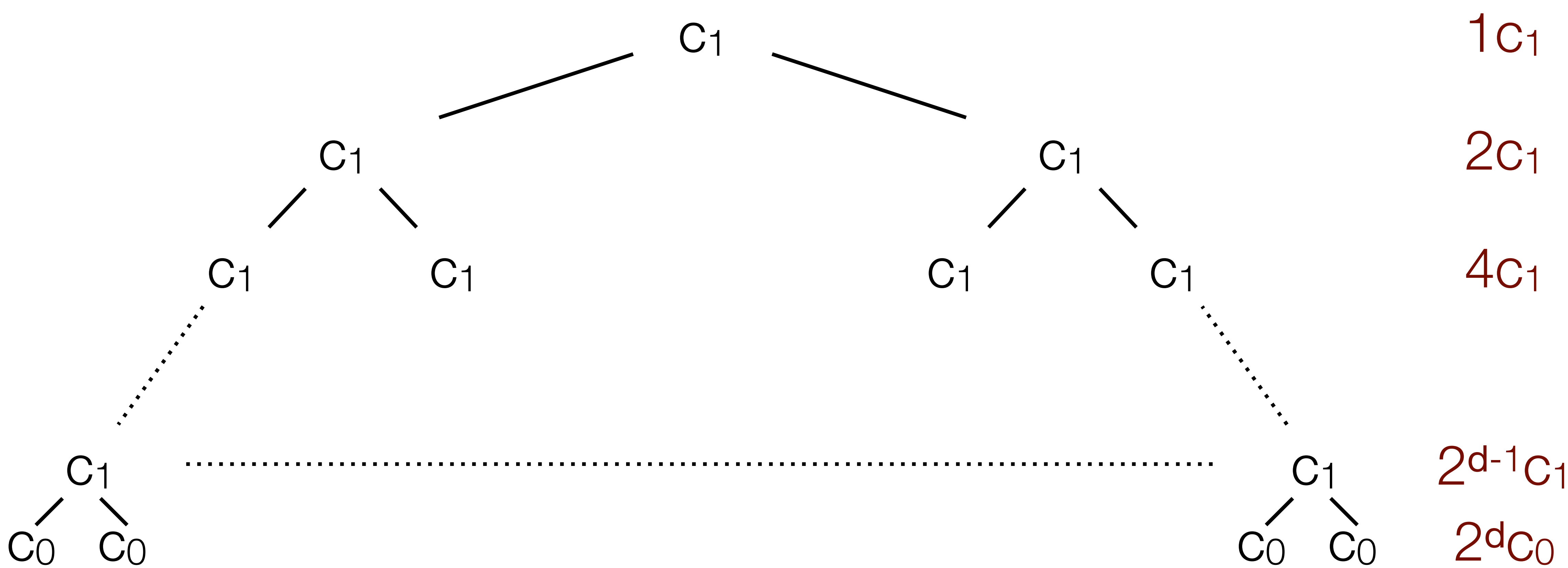
$\mathtt{W_{sum}(d)} = (2^0 + 2^1 + \ldots + 2^{d-1})\cdot \mathbf{c_1} + 2^d\cdot \mathbf{c_0} \leq 2^{d+1}\cdot \mathbf{c}$

Visualize work/span for sum (with depth)



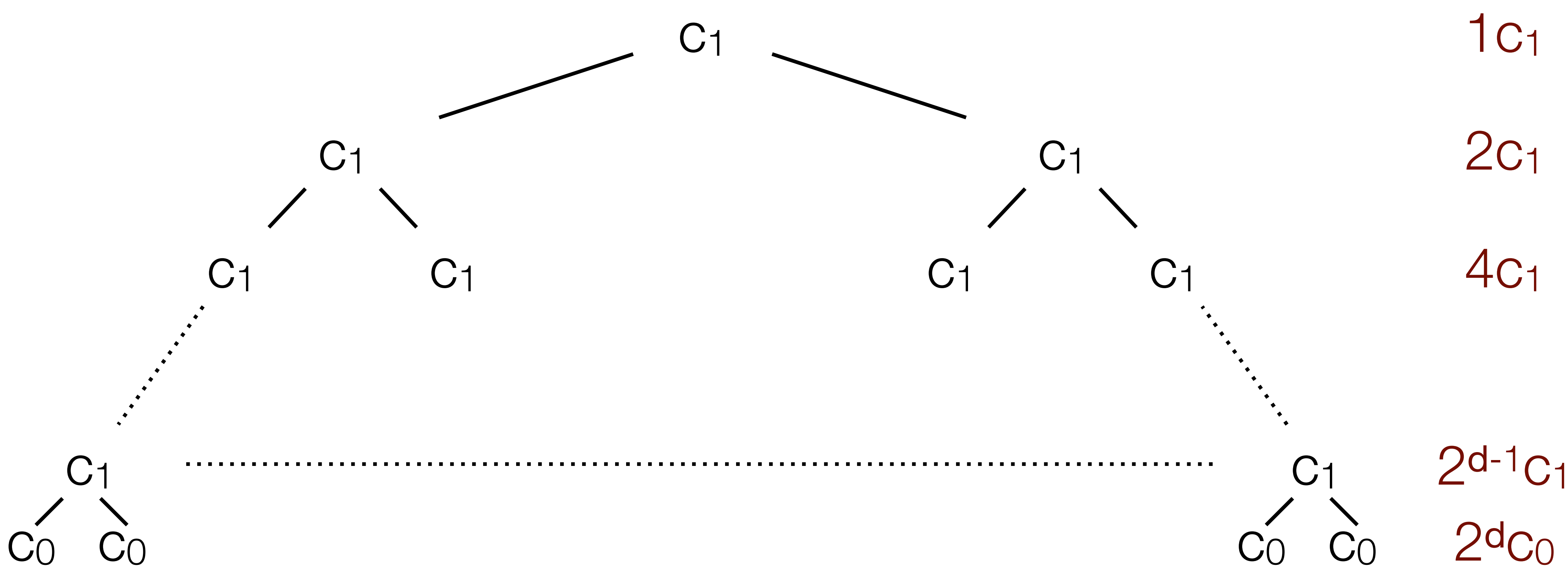
$$\mathtt{W_{sum}(d)} = (2^0 + 2^1 + \ldots + 2^{d-1})\cdot c_1 + 2^d\cdot c_0 \leq 2^{d+1}\cdot c$$

Visualize work/span for sum (with depth)



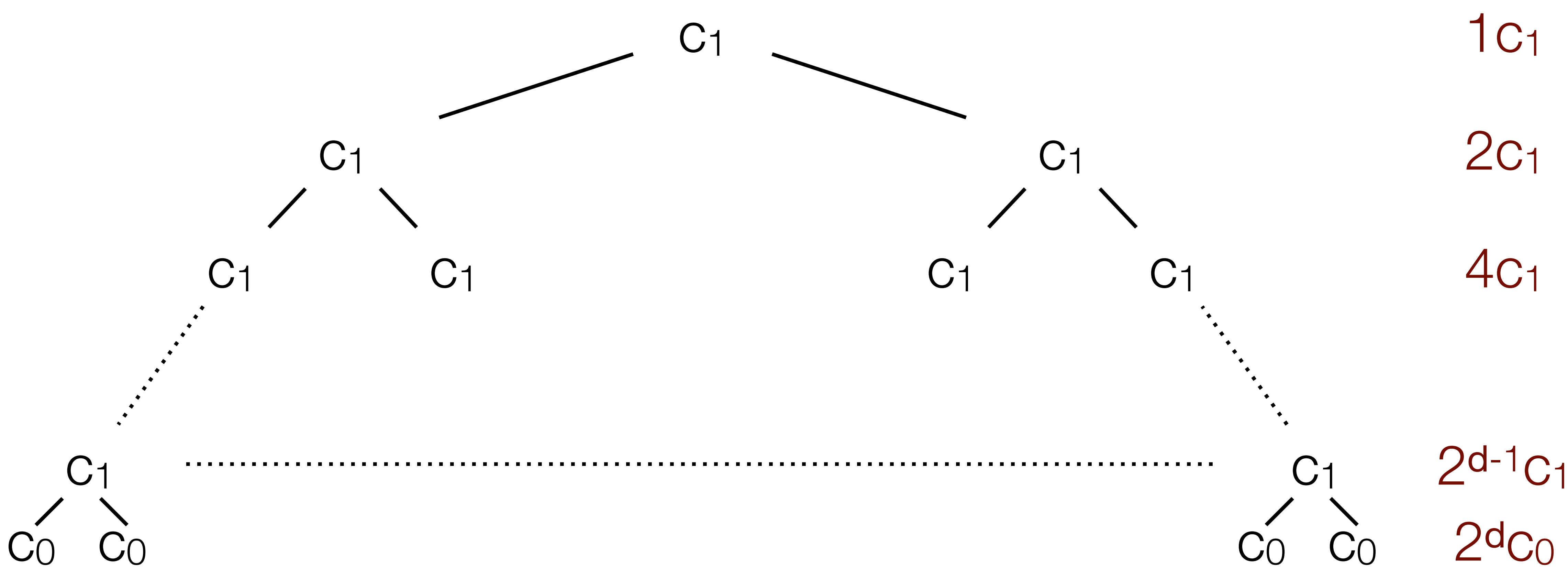
$$W_{sum}(d) = \underbrace{(2^0 + 2^1 + \ldots + 2^{d-1})}_{< 2^d} \cdot c_1 + 2^d \cdot c_0 \le 2^{d+1} \cdot c$$

Visualize work/span for sum (with depth)



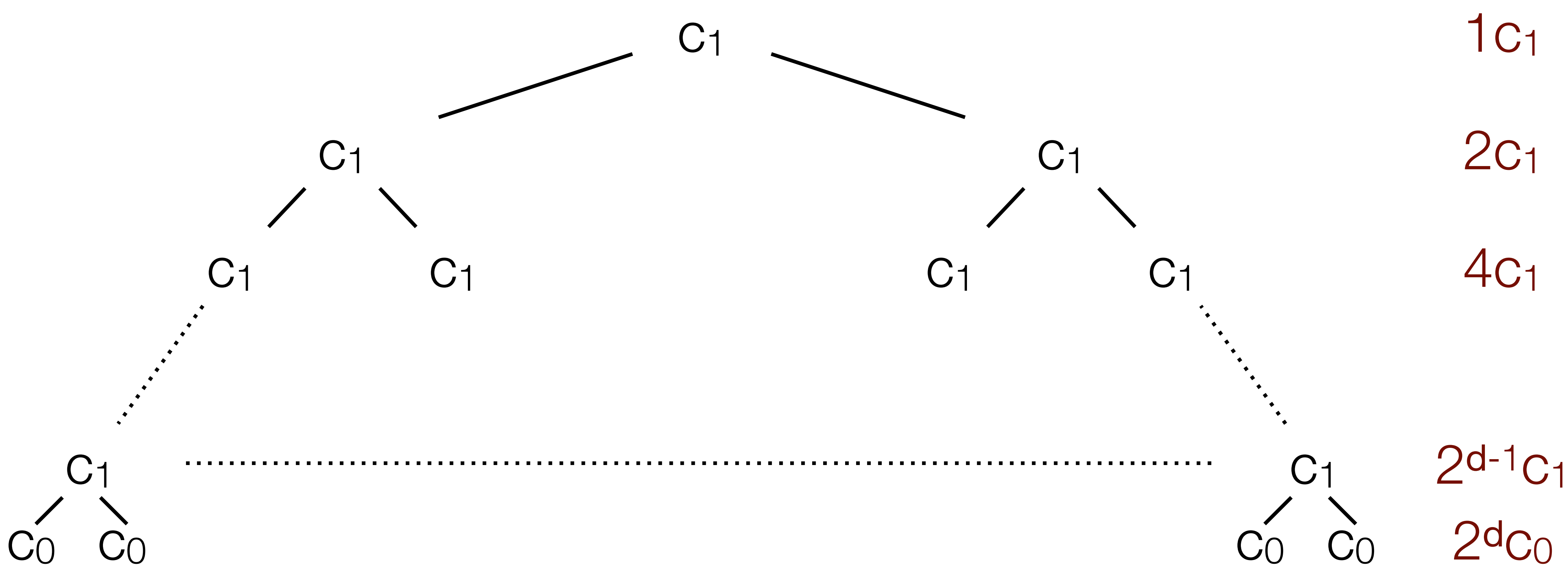
$$W_{sum}(d) = (\underbrace{2^0 + 2^1 + \ldots + 2^{d-1}}_{< 2^d})\cdot c_1 + 2^d\cdot c_0 \leq 2^{d+1}\cdot c$$

Visualize work/span for sum (with depth)



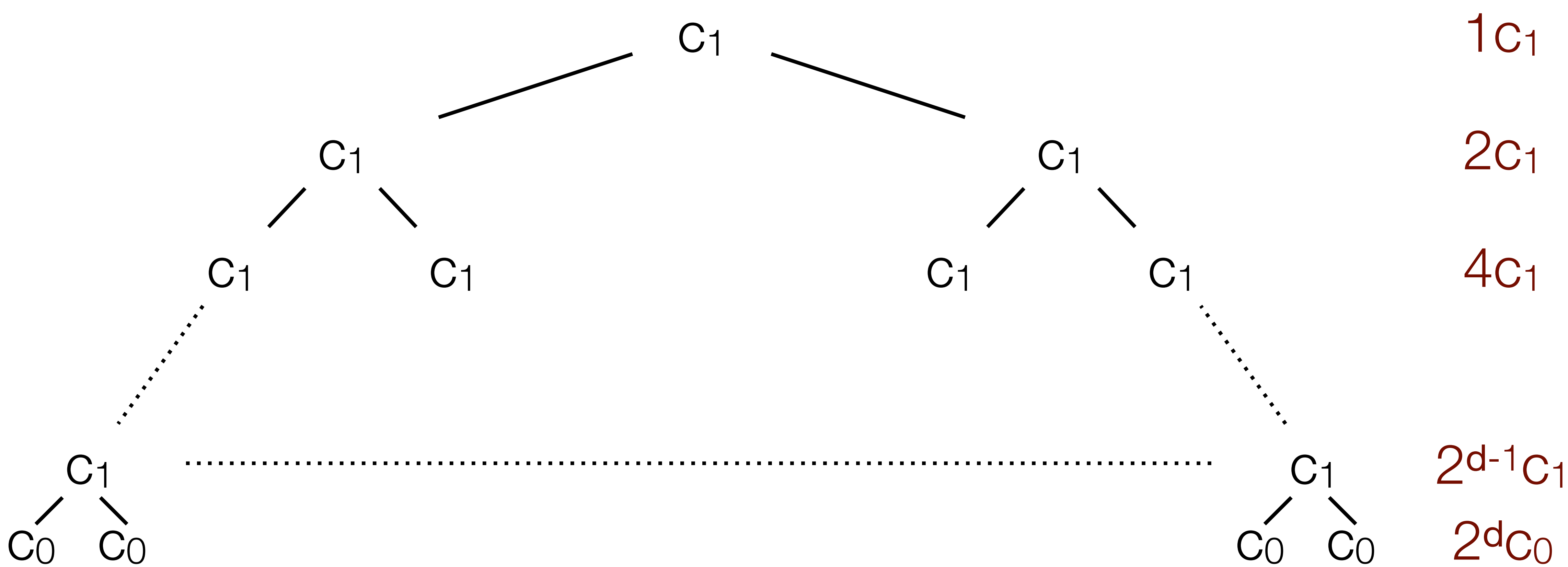
$$W_{sum}(d) = (\underbrace{2^0 + 2^1 + \ldots + 2^{d-1}}_{< 2^d})\cdot c_1 + 2^d \cdot c_0 \leq 2^{d+1} \cdot \underbrace{c}_{\max(c_0,\, c_1)}$$

Visualize work/span for sum (with depth)



$$W_{sum}(d) = (2^0 + 2^1 + \ldots + 2^{d-1})\cdot c_1 + 2^d\cdot c_0 \leq 2^{d+1}\cdot c$$

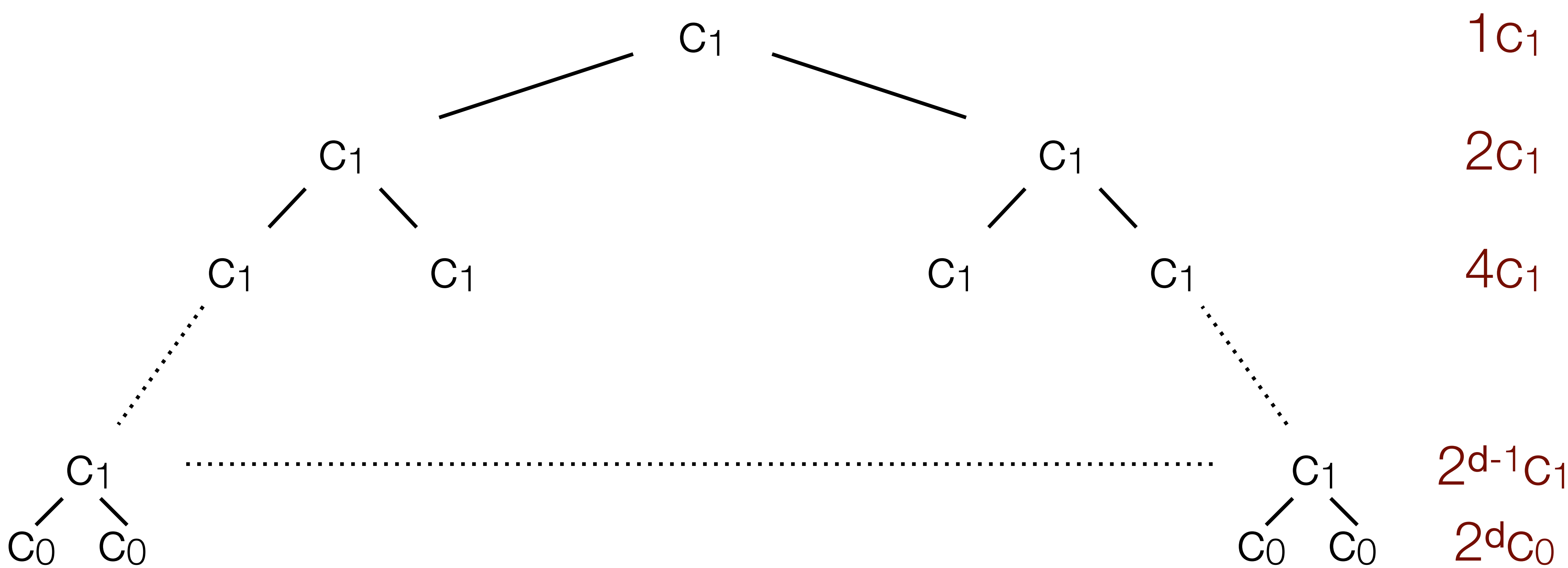
Visualize work/span for sum (with depth)



$$W_{sum}(d) = (2^0 + 2^1 + \ldots + 2^{d-1})\cdot c_1 + 2^d\cdot c_0 \leq 2^{d+1}\cdot c$$

$$S_{sum}(d) = (1 + 1 + \ldots + 1)\cdot c_1 + c_0 \leq d\cdot c$$

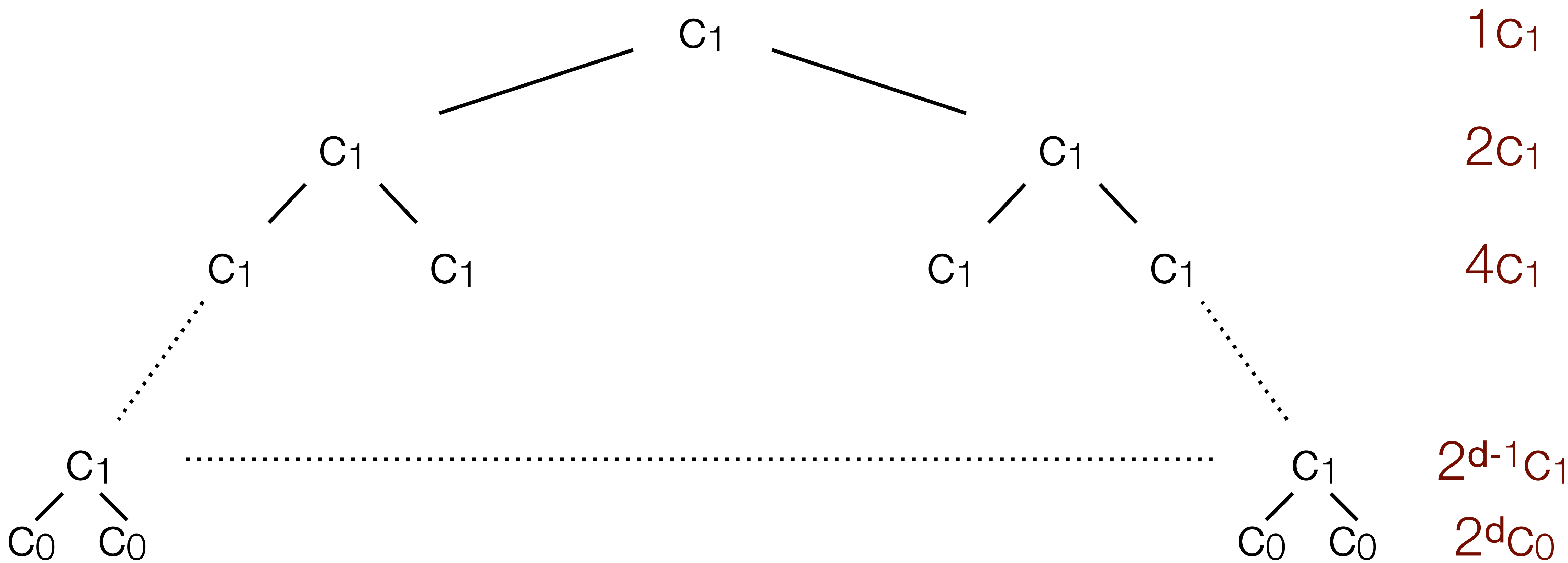
Visualize work/span for sum (with depth)



$$\texttt{W}_{\texttt{sum}}\texttt{(d)} = (2^0 + 2^1 + \ldots + 2^{d-1})\cdot c_1 + 2^d\cdot c_0 \leq 2^{d+1}\cdot c$$

$$\texttt{S}_{\texttt{sum}}\texttt{(d)} = (1 + 1 + \ldots + 1)\cdot c_1 + c_0 \leq d\cdot c$$

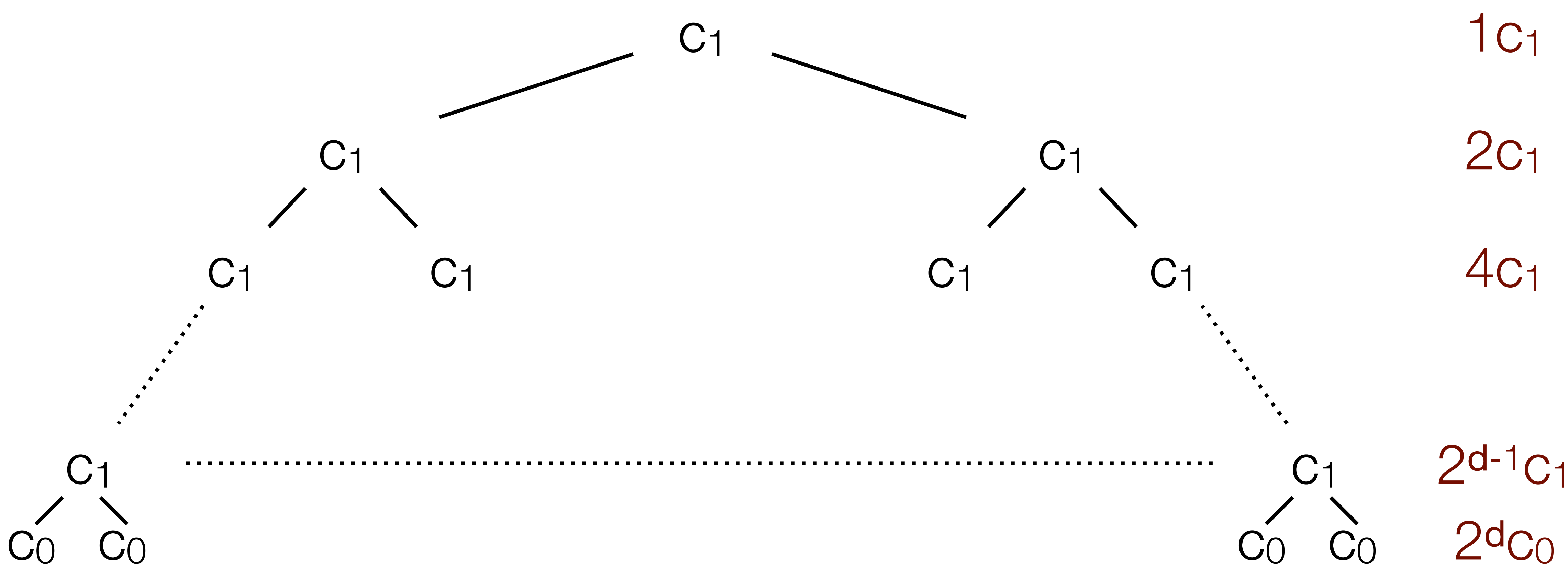
Visualize work/span for sum (with depth)



$$\mathtt{W}_{\mathtt{sum}}\mathtt{(d)} = (2^0 + 2^1 + \ldots + 2^{d-1})\cdot c_1 + 2^d\cdot c_0 \leq 2^{d+1}\cdot c$$

$$\mathtt{S}_{\mathtt{sum}}\mathtt{(d)} = (\underbrace{1 + 1 + \ldots + 1}_{\text{d-1 times}})\cdot c_1 + c_0 \leq d\cdot c$$

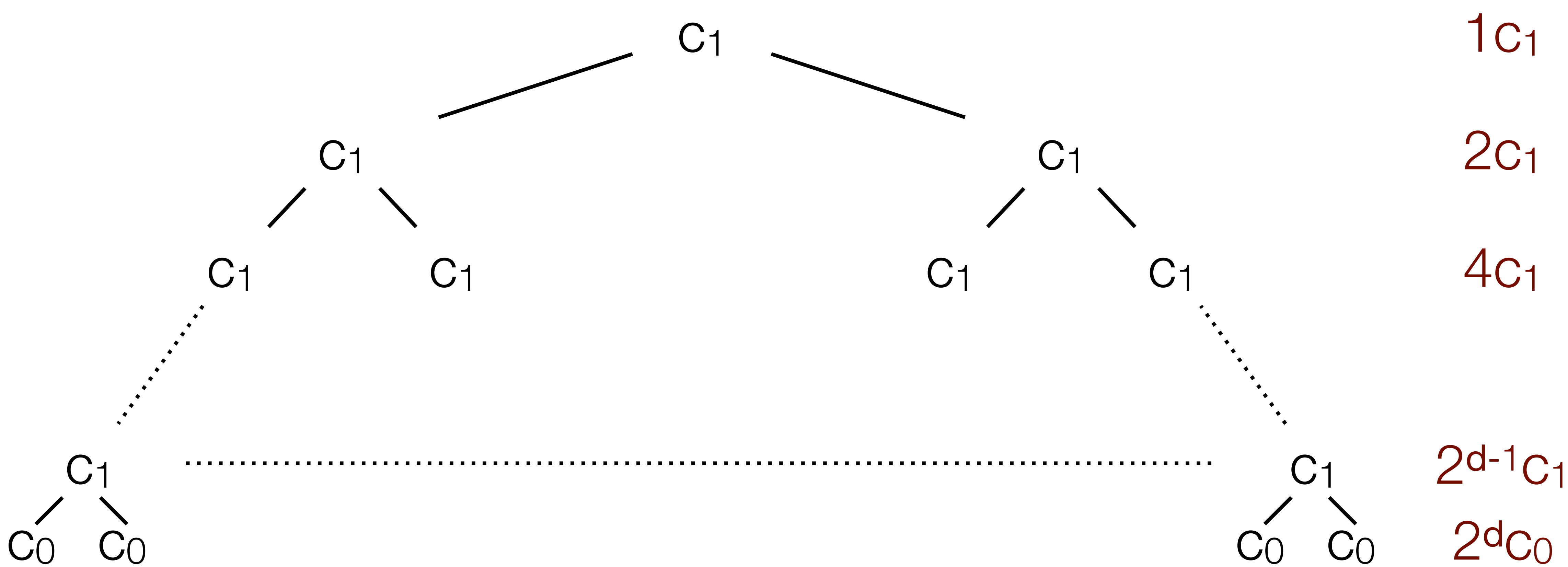
Visualize work/span for sum (with depth)



$$W_{sum}(d) = (2^0 + 2^1 + \dots + 2^{d-1})\cdot c_1 + 2^d\cdot c_0 \leq 2^{d+1}\cdot c$$

$$S_{sum}(d) = (\underbrace{1 + 1 + \dots + 1}_{\text{d-1 times}})\cdot c_1 + c_0 \leq d\cdot c$$

Visualize work/span for sum (with depth)



$$W_{sum}(d) = (2^0 + 2^1 + \ldots + 2^{d-1})\cdot c_1 + 2^d \cdot c_0 \leq 2^{d+1}\cdot c$$

$$S_{sum}(d) = (\underbrace{1 + 1 + \ldots + 1}_{\text{d-1 times}})\cdot c_1 + c_0 \leq d\cdot \underbrace{c}_{\max(c_0,\, c_1)}$$

That's all for today. See you on Thursday!