Lists, structural induction, tail recursion, and extensional equivalence

15-150

Lecture 5: September 4, 2025

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Taking stock

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more on datatypes in
later lectures!

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  - today: example using **structural induction** on lists

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can be matched against!

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cons evaluates left to right (for sequential evaluation)

Length function for an int list

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   ENSURES: length(L) returns the number of
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patterns!

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- Unlike in math, functions in SML are not total
  - eg, recursive functions can loop!

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An expression `f: X -> Y` is **total**, if `f` reduces to a value (ie function) and `f(x)` reduces to a value for all values `x` in `X`.

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→ Partiality complicates extensional equivalence (more later!)

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To prove a property P(L) for every value L of type int list:

- show that P([ ]) holds
- show that, if P(L') for some value L' of type int list,
  then it also holds for v :: L' (for any value v of type int).

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- We proceed by structural induction on argument list.
  - Let’s do the proof together!

Length is total: proof

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Theorem: For al values `L:int list`, `length(L)` evaluates to an integer value `v`.

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(step, 2nd clause of `length`)

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SML addition
must be total!

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Data structure **Code** **Proof**

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inductive/recursive case(s):

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`0`	`fun power(_, 0) = 1`	`power(_, 0)` $\hookrightarrow$ 1
inductive/recursive case(s):		
`(k-1)+1`	`power(n,k) = n*power(n, k-1)`	IH: `power(n,k)` $\hookrightarrow n^k$ NTS: `power(n,k+1)` $\hookrightarrow n^{k+1}$

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`[ ]`	`0`	`length []` ↪ 0
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`x :: xs`	`length(x::xs) = 1+length(xs)`	IH: `length(xs)` ↪ v NTS: `length(x::xs)` ↪ v’

→ possibly several bases and inductive cases!

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[…, 3/x, [4,5]/xs] 1 + length(xs)
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               ==> 1 + (1 + 1)
               ==> 1 + 2
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               ==> 1 + (1 + 1)
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               ==> 3
```

Let's reconsider our length function

```
fun length ([] : int list) : int = 0
  | length (x::xs) = 1 + length(xs)
```

Is this function efficient? Well, let's look at an evaluation trace:

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time

each recursive call has pending computations

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   ENSURES: tlength(L, acc) == length(L) + acc
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accumulator

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extensional equivalence

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space-inefficient version

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fun tlength([]: int list, acc: int): int = acc
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tail call

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```

tail call

`1+acc` happens before recursive call!

Remember: evaluation order for applications

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Function application: `e1 e2`

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Function application: `e`$_1$ `e`$_2$

1 Reduce `e`$_1$ to a function `value fn (x:t) => e`.

Remember: evaluation order for applications

Function application: `e`$_1$ `e`$_2$

1. Reduce `e`$_1$ to a function `value fn (x:t) => e`.
2. Reduce `e`$_2$ to a value `v`.

Remember: evaluation order for applications

Function application: `e`$_1$ `e`$_2$

1. Reduce `e`$_1$ to a function value `fn (x:t) => e`.
2. Reduce `e`$_2$ to a value `v`.
3. Extend function closure environment with binding for `x` and `v`.

Remember: evaluation order for applications

Function application: $\texttt{e}_1\ \texttt{e}_2$

1. Reduce $\texttt{e}_1$ to a function `value fn (x:t) => e`.
2. Reduce $\texttt{e}_2$ to a value `v`.
3. Extend function closure environment with binding for `x` and `v`.
3. Evaluate function body `e` in resulting environment.

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fun length' (L: int list): int = tlength (L, 0)
```

Improved space efficiency

```
fun tlength([]: int list, acc: int): int = acc
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length’ [3,4,5] ==> tlength ([3,4,5], 0]
                ==> tlength ([4,5], 1+0)
                ==> tlength ([4,5], 1)
                ==> tlength ([5], 1+1)
                ==> tlength ([5], 2)
                ==> tlength ([], 2+1)
                ==> tlength ([], 3)
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time

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space

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time

recursive
calls without pending
computations

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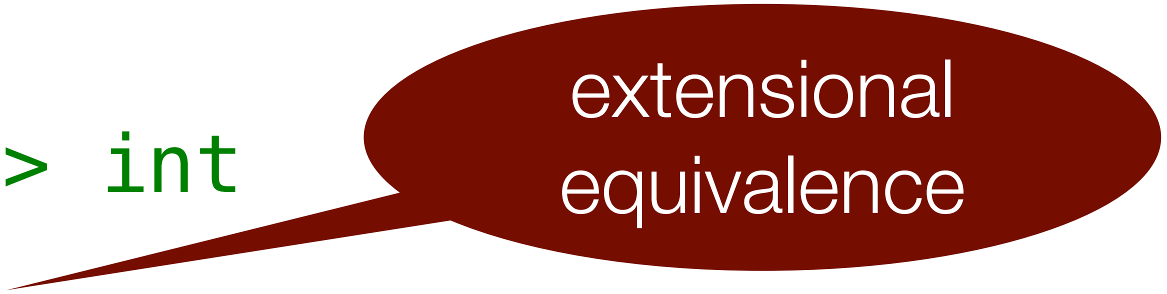

Let's prove space-efficient version correct!

extensional equivalence

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Theorem: For all values `L: int list` and `acc: int`,
`tlength(L,acc)` ≅ `(length L) + acc`.

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extensional equivalence

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Before doing the proof, let's review extensional equivalence!

Extensional equivalence, revisited

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Definition:

- **Expression** of type `int` are **extensionally equivalent**,
  - if they evaluate to the same integer, or
  - if they both loop forever, or
  - if they both raise the same exception.
- **Functions** of type `int -> int` are **extensionally equivalent**, if they yield extensionally equivalent results, **if** applied to extensionally equivalent arguments.

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Facts:

- If $e_1 \hookrightarrow v$ and $e_2 \hookrightarrow v$, then $e_1 \cong e_2$.
- If $e_1 \Longrightarrow e$ and $e_2 \Longrightarrow e$, then $e_1 \cong e_2$.
- If $e_1 \Longrightarrow e_2$, then $e_1 \cong e_2$.

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However:

From $e_1 \cong e_2$ it does **not** necessarily follow that
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eg, $1+1+7 \cong 9$

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However:

From $e_1 \cong e_2$ it does **not** necessarily follow that
$e_1 \Longrightarrow e_2$ or $e_2 \Longrightarrow e_1$!
eg, `1+1+7` $\cong$ `9`

Moreover:

For f: `int -> int`, it does **not** necessarily follow that
`f(1) + f(2)` $\cong$ `f(2) + f(1)`!

Extensional equivalence, revisited

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eg, $1+1+7 \cong 9$

f may loop or raise an exception (on some inputs)

Moreover:

For f: `int -> int`, it does **not** necessarily follow that `f(1) + f(2)` $\cong$ `f(2) + f(1)`!

Let’s prove space-efficient version correct!

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fun length ([] : int list) : int = 0
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fun tlength([]: int list, acc: int): int = acc
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```

Theorem: For all values `L: int list` and `acc: int`, `tlength(L,acc)` $\cong$ `(length L) + acc`.

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Theorem: For all values `L: int list` and `acc: int`, `tlength(L,acc)` ≅ `(length L) + acc`.

Proof: By ???

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Theorem: For all values `L: int list` and `acc: int`,
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Proof: By structural induction on L.

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Proof: By structural induction on L.

Base case: L = `[]`.

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Need to show: for all values `acc:int`,

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Evaluating the left side:

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Evaluating the left side:

```
    tlength ([], acc)
⟹  acc
```

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Need to show: for all values `acc:int`,

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Evaluating the left side:

`tlength ([], acc)`

⟹ `acc` (1st clause of `tlength`)

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Showing:

Evaluating the left side:

$\quad$ `tlength ([], acc)`

$\Longrightarrow$ `acc` (1st clause of `tlength`)

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fun length ([] : int list) : int = 0
  | length (x::xs) = 1 + length(xs)

fun tlength([]: int list, acc: int): int = acc
  | tlength(x::xs, acc) = tlength(xs, 1 + acc)
```

Showing:

Evaluating the left side:

```
    tlength ([], acc)
⟹  acc
```

(1st clause of `tlength`)

Evaluating the right side:

Let's prove space-efficient version correct!

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fun length ([] : int list) : int = 0
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<u>Showing</u>:

Evaluating the left side:

    tlength ([], acc)
$\Longrightarrow$ `acc` (1st clause of `tlength`)

Evaluating the right side:

    length([]) + acc

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`length([]) + acc`

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$\Longrightarrow$	`acc`	(SML’s arithmetic)

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Evaluating the right side:

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$\Longrightarrow$ `acc` (SML's arithmetic)

equivalent because both reduce to the same value.

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<u>IH</u>: for all values `acc’:int`,

`tlength(xs,acc’)` $\cong$ `length (xs) + acc’`.

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Showing:

`tlength(x::xs, acc)`

$\cong$ `tlength(xs, 1 + acc)` (step, 2nd clause of `tlength`)

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	`tlength(x::xs, acc)`	
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≅	`length(xs) + (1 + acc)`	(IH, assume + total)

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because: if $e_1 \Longrightarrow e_2$, then $e_1 \cong e_2$

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Proved by structural induction on lists.

Appending lists

Appending lists

```
(* append: int list * int list -> int list
   REQUIRES: true
   ENSURES: append(L,R) evaluates to a list consisting
            of L followed by R
   NOTE: predefined in SML as the right-associative
         infix operator @.
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Appending lists

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val [1,2] = append([], [1,2])
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What is the time complexity of append?

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What is the time complexity of append?

append(X,Y) has time complexity O(|X|).

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val [4,3,2,1] : int list = rev [1,2,3,4]
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reverse(X) has time complexity $O(|X|^2)$.

Reversing a list: can we do better?

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accumulator

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Are reverse and rev extensional equivalent?

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Theorem: For all values `L: int list` and `acc: int list`, `trev(L,acc)` $\cong$ `(rev L) @ acc`.

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- Prove this theorem as an exercise!
- We provide the solution in the notes (rev.pdf). But try first!

That's all for today. Have a good weekend!