15-150
Fall 2025

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LECTURE 2

Functions

Last time

- Types, expressions, values
- Extensional equivalence
- Declaration

Today’s Goals

- Write anonymous functions (lambda expressions)
- Declare named functions
- State what a function closure is
- Evaluate expressions that involve function application
- Use patterns
  - clausal function declarations
  - case expressions

Declarations, Environments, Scope

Declaration

```
val pi : real = 3.14
```

keyword identifier type value

Introduces binding of `3.14` to `pi`, sometimes written as [`3.14/pi`]

Lexically statically scoped

Bindings

```
val x : int = 8 - 5     [3/x]
val y : int = x + 1     [4/y]
```

Bindings

```
val x : int = 8 - 5     [3/x]
val y : int = x + 1     [4/y]
val x : int = 10        [10/x]
val z : int = x + 1     [11/z]
```

Second binding of x ***shadows*** first binding. First binding has been *shadowed*.

Local declarations

```
let
  val m : int = 3
  val n : int = m * m
in
  m + n
end
```

This is an expression with type `int` and value `12`.

Local declarations

```
val k : int = 4

let
  val k : real = 3.0
in
  k * k
end
```

Type?
Value?

Local declarations

```
val k : int = 4

let
  val k : real = 3.0
in
  k * k
end
```

Type?
Value?

9.0 : real

```
k
```

Type?
Value?

4 : int

Concrete Type Definitions

```
type float = real
type point = float * float
val p : point = (1.0, 2.6)
```

```
val p : point = (1.0, 2.6)
```

creates the binding `[(1.0, 2.6)/p]`

Functions

Function declaration

```
(* square : int -> int
   REQUIRES: true
   ENSURES: square(x) evaluates to x * x
*)

fun square (x : int) : int = x * x
```

function name

function body

Function declaration

```
fun square (x : int) : int = x * x;
```

creates the binding `[???/square]`

Closures

```
fun square (x : int) : int = x * x
```

binds the identifier **`square`** to a closure:

```
[ /square]
```

code for square, lambda expression **`fn`** `(x:int) => x * x`
environment (all prior bindings when square was declared)

Anonymous functions
a.k.a. lambda

```
fn (x : int) => x * x
```

formal parameter

argument type

body

Functions are values

```
fn (x : int) => x * x
```

This expression is already a value —
no further evaluation happens

Alternative way to declare

```
fun square (x : int) : int = x * x;
```

is syntactic sugar for

```
val square = fn (x : int) => x * x
```

Applying a function

$$\underbrace{\texttt{(fn (x : int) => x * x)}}_{\textbf{function}}\ \underbrace{\texttt{7}}_{\textbf{argument}}$$

This function application evaluates to 49.

Applying a function

$$\underbrace{\texttt{(fn (x : int) => x * x)}}_{e_1}\ \underbrace{\texttt{7}}_{e_2}$$

How does ML evaluate a function application $e_1\ e_2$?

- Evaluate e_1 to a function value of the form

  `fn(x : t) => e`

- Reduce e_2 to a value v
- Locally extend the environment that existed at the time of the definition of function with a binding of value v to the variable x
- Evaluate the body e in the resulting environment

Function types

Function types are of the form

$$t_1 \texttt{ -> } t_2$$

argument   result

Typing rules

```
(fn (x : t₁) => e): t₁ -> t₂
```

if e: t_2 assuming x : t_1

Examples:

```
(fn (x : int) => x): int -> int
(fn (x : real) => x): real -> real
```

Typing rules

`(`**`fn`**` (x : `t_1`) => e): `t_1` -> `t_2

if `e:` t_2 assuming `x :` t_1

`e`$_1$ `e`$_2$ `:` t_2 if `e`$_1$`:` t_1 `->` t_2 and `e`$_2$ `:` t_1

Function closures

Environment together with a function expression

```
[3.14/pi](fn (r : real) => pi * r * r)
```

The environment provides the
bindings of nonlocal variables

Static scope

```
val pi : real = 3.14

fun area (r:real):real = pi * r * r
```

Function Application

```
val pi : real = 3.14

fun area (r:real):real = pi * r * r
```

What does SML do with the following expression?

```
area (1.9 + 2.1)
```

Answer: First type-check, if well-typed then evaluate

Type-check

`area (1.9 + 2.1)`

`area: real -> real`

because `area` is `(fn (r : real) => pi * r * r)`

and `pi * r * r : real`

given that `pi: real` by its declaration

and `r: real` by type annotation

$\boxed{\texttt{area (1.9 + 2.1)}}$ `:real`

Evaluate `area (1.9 + 2.1)`

```
   area (1.9 + 2.1)
==>[3.14/pi](fn (r : real) => pi * r * r)(1.9 + 2.1)
==>[3.14/pi](fn (r : real) => pi * r * r)(4.0)
==>[3.14/pi][4.0/r] (pi * r * r)
==> 50.24
```

Also written as

$\boxed{\texttt{area (1.9 + 2.1)} \hookrightarrow \texttt{50.24}}$

Shadowing

```
val pi : real = 3.14

fun area (r:real):real = pi * r * r


area 1.0 ↪ 3.14
val pi : real = 3.14519
area 1.0 ↪ ???
```

Shadowing

```
val pi : real = 3.14

fun area (r:real):real = pi * r * r


area 1.0 ↪ 3.14
val pi : real = 3.14519
area 1.0 ↪ 3.14
```

Shadowing

```
val pi : real = 3.14

fun area (r:real):real = pi * r * r


area 1.0 ↪ 3.14
val pi : real = 3.14519

fun new_area(r:real) : real = pi * r * r
area 1.0 ↪ 3.14
new_area 1.0 ↪3.14519
```

Example: 5-step methodology

```
(* fact : int -> int
   REQUIRES:  n >= 0
   ENSURES: fact(n)  = n!
*)

fun fact(0: int): int = 1
  | fact(n: int): int = n * fact(n-1)
```

Example: 5-step methodology

```
(* fact : int -> int
   REQUIRES:  n >= 0
   ENSURES: fact(n)  = n!
*)

fun fact(0: int): int = 1
  | fact(n: int): int = n * fact(n-1)

fact 0
fact 6
```

pattern matching determines which clause is evaluated

Patterns

- Constant (e.g. 0, “and”, no reals)
- Variable (e.g. n)
- Tuple of patterns (p_1, …, p_n)
- Based on user-defined datatypes
- The wildcard _

Patterns

Matching

- A *pattern* can be *matched* against a *value*
- If the match *succeeds,* it produces *bindings*

Patterns in function declarations

```
fun    f p1 = e1
      |f p2 = e2
       …
      |f pk = ek
```

Patterns in variable declarations

```
val p = e
```

```
val (k, r) : int * real =  (2, 3.14)
```

This declaration creates two variable bindings:

```
[2/k, 3.14/r]
```

Constant patterns

```
val p = e
```

```
val 49 = square (7)
```

Succeeds only if the value returned by `square` is 49.

fib

n	0	1	2	3	4	5
f_n	1	1	2	3	5	8

```
(* fib : int -> int
   REQUIRES: n >= 0
   ENSURES: fib(n) ==> fₙ (nth Fibonacci number)
 *)
```

```
fun fib (0 : int) : int = 1
  | fib (1 : int) : int = 1
  | fib (n : int) = fib (n-1) + fib (n-2)
```

Efficient `fib`

n	0	1	2	3	4	5
$\mathbf{f_n}$	1	1	2	3	5	8

```
(* fibb : int -> int * int
   REQUIRES: n >= 0
   ENSURES: fibb(n) ==> (fn,fn-1)
   with fn the nth Fibonacci number (let f-1 = 0)
 *)
```

Efficient `fib`

n	0	1	2	3	4	5
f_n	1	1	2	3	5	8

```
(* fibb : int -> int * int
   REQUIRES: n >= 0
   ENSURES: fibb(n) ==> (fn,fn-1)
   with fn the nth Fibonacci number (let f-1 = 0)
 *)

 fun  fibb (0:int):int * int = (1,0)
    | fibb 1 = (1,1)
    | fibb n = let
                 val
                in

                end
```

Efficient `fib`

n	0	1	2	3	4	5
f_n	1	1	2	3	5	8

```
(* fibb : int -> int * int
   REQUIRES: n >= 0
   ENSURES: fibb(n) ==> (fn,fn-1)
   with fn the nth Fibonacci number (let fn-1 = 0)
 *)

 fun  fibb (0:int):int * int = (1,0)
    | fibb 1 = (1,1)
    | fibb n = let
                 val (f1:int,f2:int) = fibb (n-1)
               in

               end
```

Efficient `fib`

n	0	1	2	3	4	5
f_n	1	1	2	3	5	8

```
(* fibb : int -> int * int
   REQUIRES: n >= 0
   ENSURES: fibb(n) ==> (fₙ,fₙ₋₁)
   with fₙ the nth Fibonacci number (let fₙ₋₁ = 0)
 *)

 fun  fibb (0:int):int * int = (1,0)
    | fibb 1 = (1,1)
    | fibb n = let
                  val (f1:int,f2:int) = fibb (n-1)
                in
                  (f1 + f2, f1)
                end
```

Using patterns in `case` expressions

```
fun fact(n:int):int =
  case n of
       0 => 1
     | _ => n * fact(n-1)
```

Patterns in case expressions

```
(case e : t' of
            p1 => e1
          | …
          | pk => ek)
```

How do we find the type of this case expression?

Example: case

```
(* example: int -> int
   REQUIRES: true
   ENSURES: example(x) returns 0 if x = 1,
                               x * x - 1 if x < 1,
                   and 1 - x * x * x  if x > 1.
*)
```

```
 fun example (x:int):int =
      (case (square x, x > 0) of
          (1, true) => 0
        | (sqr, false) => sqr - 1
        | (sqr, true) => 1 - x * sqr)
```

Functions as first-class values

```
(* square : int -> int
   REQUIRES: true
   ENSURES: square(x) evaluates to x * x
*)

fun square (x : int) : int = x * x
```

```
(* sqrf : (int -> int) * int -> int)
   REQUIRES: true
   ENSURES: sqrf (f,x) ==> (f(x) * f(x))
 *)
```

```
fun sqrf  (f: int -> int, x: int): int = square (f(x))
```

```
val 36 = sqrf ((fn (n:int) => n+2),4)
```

```
(* sqrf : (int -> int) * int -> int)
   REQUIRES: true
   ENSURES: sqrf (f,x) ==> (f(x) * f(x))
 *)


fun sqrf  (f: int -> int, x: int): int = square (f(x))


val 36 = sqrf ((fn (n:int) => n+2),4)


fun dotwice (f: int -> int, x: int): int =
```

```
(* sqrf : (int -> int) * int -> int)
   REQUIRES: true
   ENSURES: sqrf (f,x) ==> (f(x) * f(x))
 *)


fun sqrf  (f: int -> int, x: int): int = square (f(x))


val 36 = sqrf ((fn (n:int) => n+2),4)


fun dotwice (f: int -> int, x: int): int =
                              sqrf (fn (n:int) => sqrf(f,n), x)


dotwice (fn (k:int ) => k, 3)
```

↪ ???

identity function

Some comments about $\cong$

- If $e_1 \hookrightarrow v$ and $e_2 \hookrightarrow v$ then $e_1 \cong e_2$.
- If $e_1 \texttt{==>} e_2$, then $e_1 \cong e_2$.
- If $e_1 \texttt{==>} e$ and $e_2 \texttt{==>} e$, then $e_1 \cong e_2$.
- Caution: $e_1 \cong e_2$ does not necessarily mean $e_1 \texttt{==>} e_2$ *or* $e_1 \texttt{==>} e_2$

```
Example: 1+1+1+7 ≅ 2+8
```

Some comments about ≅

`[3/y,5/z] (fn (x:int) => x + y + z)` ≅ `(fn (x:int) => x + 8)`

Totality

- Issue: Computationally a function may not always return a value. This complicates equivalence checking.

Definition: A function `f : X -> Y` is *total* if `f` reduces to a value and `f(x)` reduces to a value for all values `x` in `X`.

```
fun fact(0: int): int = 1
  | fact(n: int): int = n * fact(n-1)
```

Is fact total?

Some comments about totality

Suppose `f: int -> int`

If f is total,

`f(1) + f(2)` $\cong$ `f(2) + f(1)`