Runtime and Big-O Notation

15-110 - Wednesday 2/26

Learning Objectives

- Define the concepts of **efficiency**, **runtimes**, **function families**, and **Big-O notation**
- Compare the **function families** that different functions run in
- Identify the **worst case** and **best case** of functions
- Calculate a specific function's efficiency using **Big-O notation**

Efficiency = Time = Money

We'll talk about efficiency a lot in this unit. Why do we care?

Computers are fast, but they can still take time to do complex actions. And people don't like to wait. A faster algorithm can lead to a company succeeding where others fail.

A major goal of computer scientists is not just to make algorithms that work, but algorithms that work **efficiently**.

Linear Search vs. Binary Search

Comparing Linear vs. Binary Search

Recall when we raced linear search vs. binary search in the previous lecture. How can we compare these two algorithms at a more **abstract** level?

We could run both on the same input and time them. However, how quickly a program runs varies based on lots of factors (the implementation, the machine, which other programs are running, etc.)

Instead, we'll count the number of **actions** the program takes on a given input.

Counting the number of actions

What actions might we count? Recall that some actions, such as adding two numbers, become several steps when translated to bytecode. And some actions may take longer than other ones to execute.

Instead of trying to count every action, we count how many times the algorithm processes each element in the input.

For example, in linear or binary search, we often choose to count the total number of comparisons that the algorithms makes to find an item.

Linear vs. Binary Search: Search for 66

```
def linSearch(lst, item):
  if len(lst) == 0:
    return False
  elif lst[0] == item:
    return True
  else:
    return linSearch(lst[1:], item)
```

```
def biSearch(lst, item):
  if lst == [ ]:
    return False
  else:
    mid = len(lst) // 2
    if lst[mid] == item:
      return True
    elif item < lst[mid]:
      return biSearch(lst[:mid], item)
    else: # lst[mid] < item
      return biSearch(lst[mid+1:], item)
```

How many list elements are compared to 66?

linear search: 9 times
binary search: 4 times

							1st	4th	3rd		2nd			
12	25	32	37	41	48	58	60	66	73	74	79	83	91	95

Best Case and Worst Case

To truly compare the algorithms, it isn't enough to test them on a random example. We want to know how they'll do in the **best case**, and in the **worst case**, based on the inputs.

Best case: an input of size n that results in the algorithm taking the **least steps possible**.

Worst case: an input of size n that results in the algorithm taking the **most steps possible**.

Best Case and Worst Case – Linear Search

What's the **best case** for linear search?

Answer: a list where the item we search for is in the first position

What's the **worst case** for linear search?

Answer: a list where the item we search for is not in the list.

Best Case and Worst Case – Binary Search

You do: what's the **best case** input for binary search?

Answer: TBD

What about the **worst case** for binary search?

Answer: a list where the item we're searching for doesn't occur

Best Case/Worst Case Actions

How many actions do we perform in the **best case**?

For both linear search and binary search, there's just **one comparison** – the list (of any length) for which it finds the item with the first comparison.

How many actions in the **worst case**?

In linear search, we have to check **every single element**. So if the list has n elements, we do **n comparisons**.

What about binary search?

Worst Case Action Count – Binary Search

Each recursive call to binary search compares one item of the list. How many recursive calls do we make to binary search for different length lists?

List size	Number of recursive calls
1	1
2^2-1 = 3	2
2^3-1 = 7	3
2^4-1 = 15	4
2^5-1 = 31	5
2^k - 1	k
n	$\log_2(n)$

When the input length doubles, **linear search** does **twice** as many comparisons.

But, when the input length doubles, **binary search** does **just one more comparison!** Amazing!

Big-O Notation

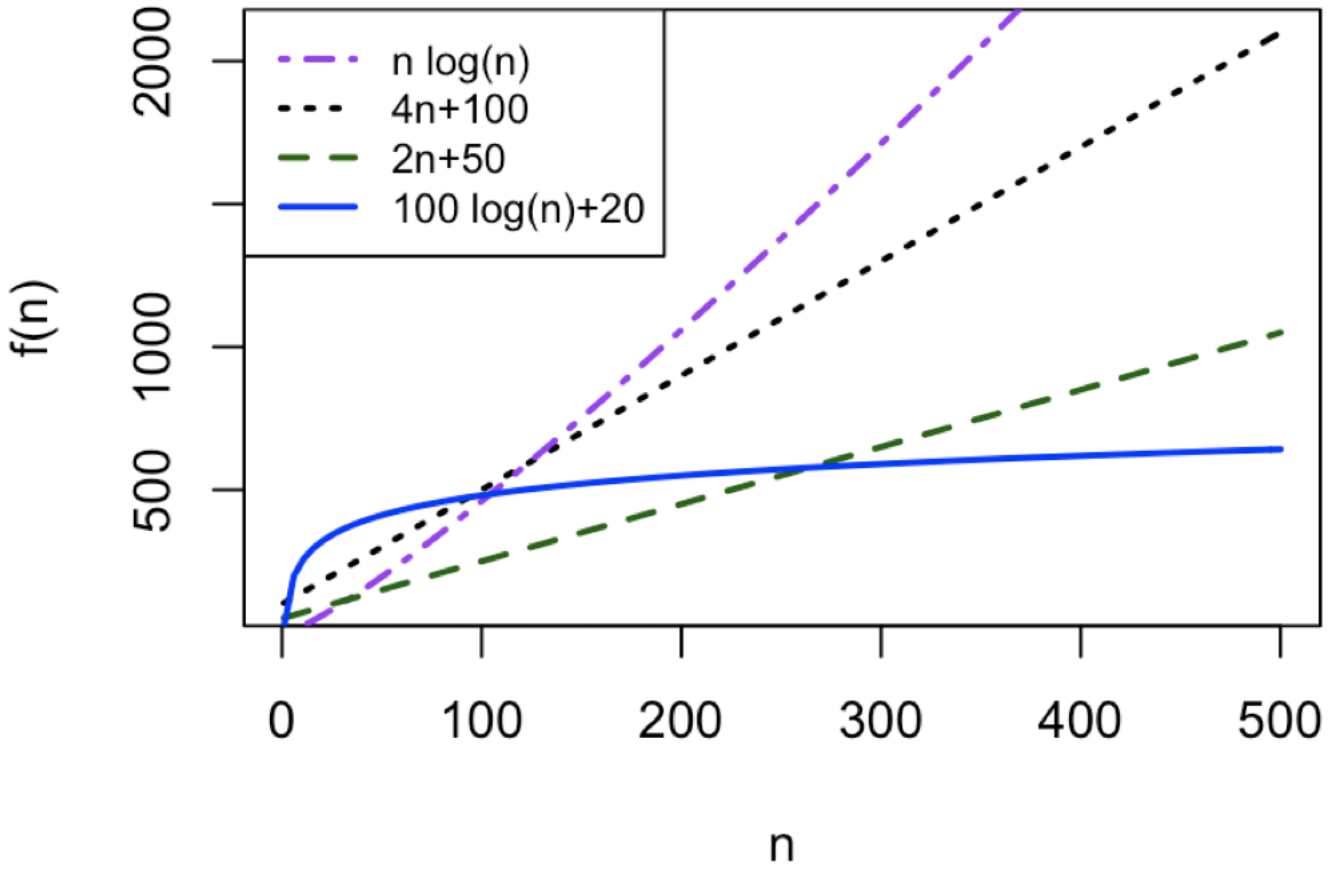
Function Families

When we count the actions taken by algorithms, we don't really care about one-off operations; we care about actions that are related to the **size of the input**.

In math, a **function family** is a set of equations that all grow at the same rate as their inputs grow. For example, an equation might grow linearly or quadratically.

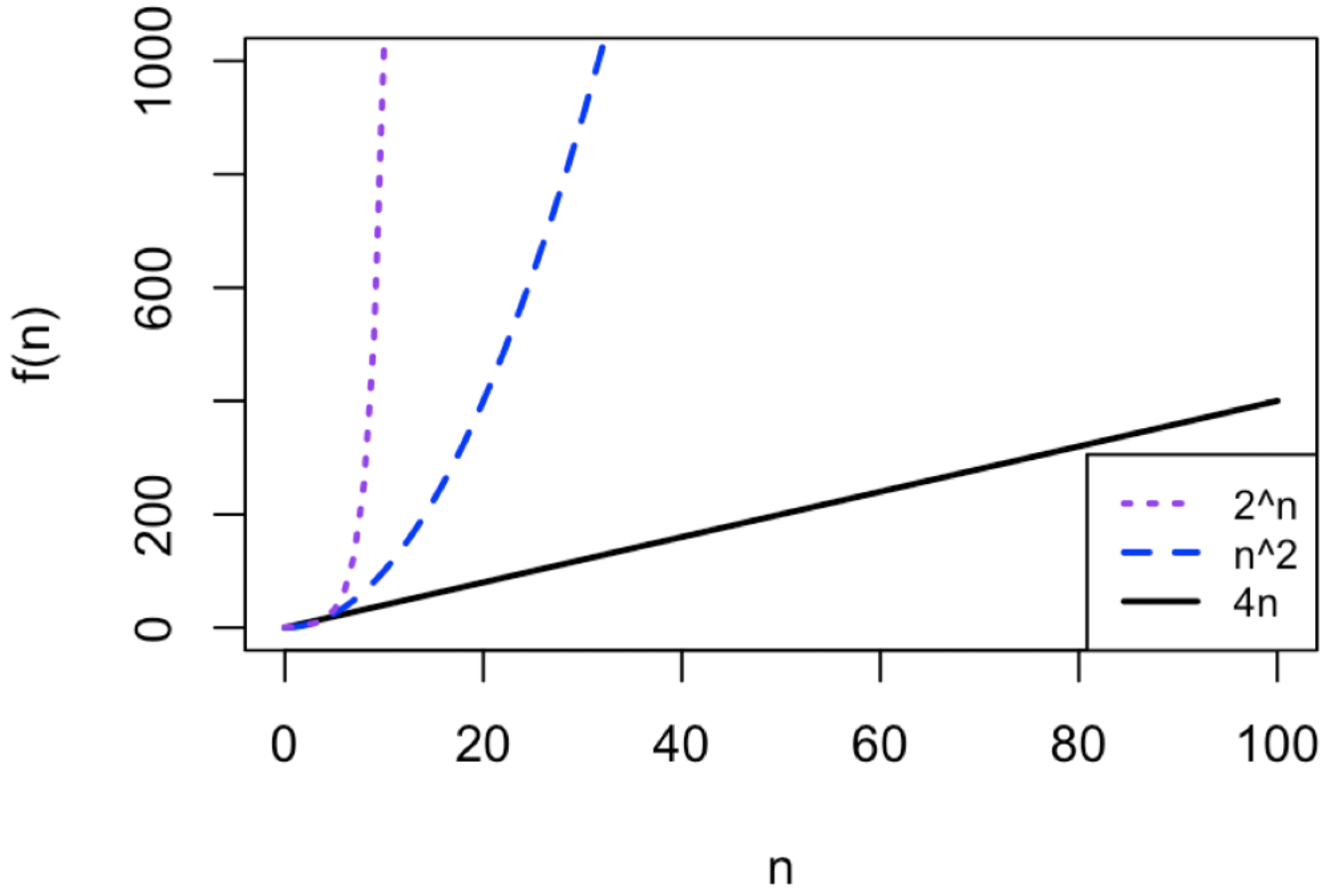
When determining which equation family represents the actions taken by an algorithm, we say that **n** is the **size of the input**. For a list, that's the number of elements; for a string, the number of characters.

Function Families in Graphs



Notice that as n grows, the two linear functions become larger than the log(n) function, and the n log(n) function becomes larger than both linear functions, regardless of the constants.

Function Families in Graphs



Even for small n, exponential functions quickly skyrocket and quadratic functions grow rapidly compared to linear functions.

Big-O Notation

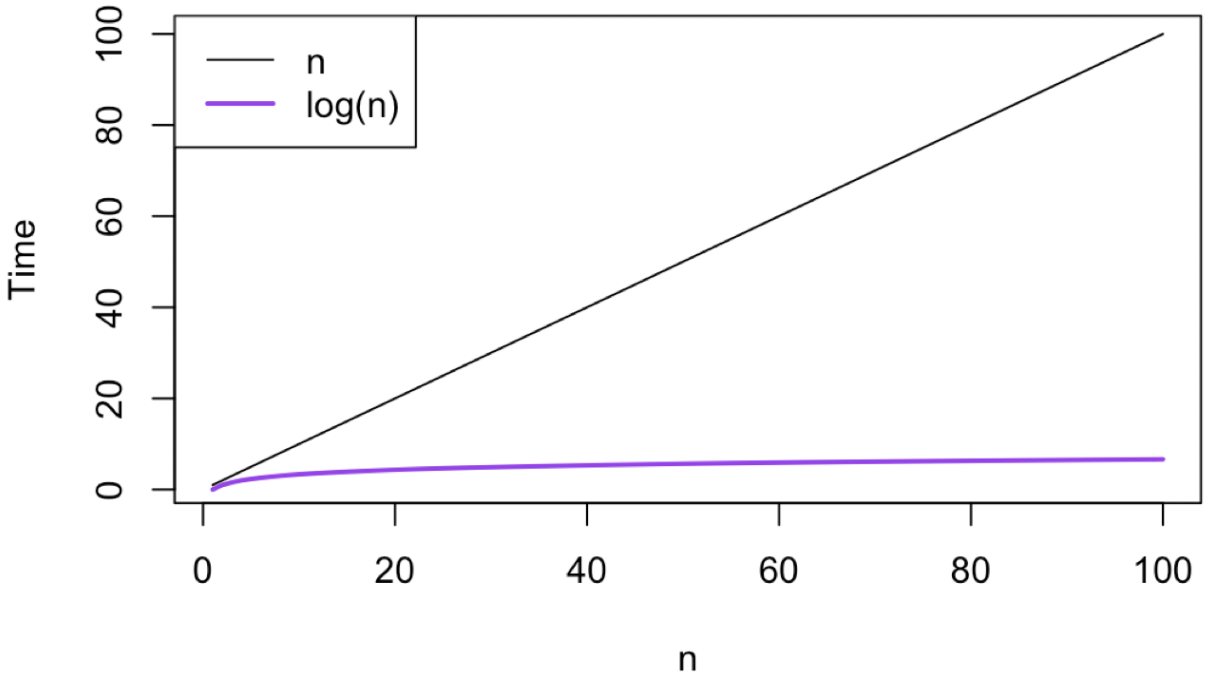
When we determine an equation's function family, we **ignore constant factors and smaller terms**. All that matters is the dominant term (the highest power of n). That is the idea of **Big-O notation.**

f(n)	Big-O
n	O(n)
32n + 23	O(n)
$5n^2 + 6n - 8$	$O(n^2)$
18 log(n)	O(log n)

Unless specified otherwise, the Big-O of an algorithm refers to its **worst case** run time (computer scientists are pessimists).

Big-O of Linear Search / Binary Search

Because runtime for linear search is proportional to the length of the list in the worst case, it is O(n). Every time we double the length of the list, binary search does just one more comparison; it is O(log n).



Except for very small n, binary search is blazingly fast. Linear search is exponentially slower in the worst case!

Big-O Calculation

Big-O Calculation Strategy

We'll often need to calculate the Big-O of an algorithm or a piece of code, to determine how efficient it is, and whether we can make it better.

We can determine a function's Big-O by determining how many actions are **added** if we increase the size of the input.

Let's go through a bunch of examples to demonstrate.

O(1) is Constant Time

```
def swap(lst, i, j):
    tmp = lst[i]
    lst[i] = lst[j]
    lst[j] = tmp
```

Does the runtime of this algorithm depend on the number of items in the list?

Answer: No.

We say that an algorithm is **constant time** when its time does not change with the size of the input.

O(log n) is Logarithmic Time

```
def countDigits(n):
    count = 0
    while n > 0:
        n = n // 10
        count = count + 1
    return count
```

Every time you increase n by a factor of 10, you do the loop one more time. All the operations in the loop are constant time. Analogous to binary search, the algorithm is O(log n).

Even though it is $\log_{10}(n)$, we don't include the base in the Big-O notation because a change of base is just a multiplicative factor.

O(n) is Linear Time

```
def countdown(n):
    for i in range(n, -1, -5):
        print(i)
```

If we double the size of n, how many more times do we go through the loop?

Answer: We double the number of times through the loop. That is linear time, as it is proportional to the size of n. Note that stepping by 5 doesn't change that it is O(n).

$O(n^2)$ is Quadratic Time

```
def multiplicationTable(n):
    for i in range(1, n+1):
        for j in range(1, n+1):
            print(i, "+", j, "=", i*j)
```

If we double the size of n, we execute the outer loop twice as many times. And for each time we execute the outer loop, we execute the inner loop twice as many times. Generating the table takes 4 times as long.

If we triple the size of n, generating the table takes 9 times as long. The runtime is proportional to n^2.

$O(2^n)$ is Exponential Time

```
def move(start, tmp, end, num):
    if num == 1:
        return 1
    else:
        moves = 0
        moves = moves + move(start, end, tmp, num - 1)
        moves = moves + move(start, tmp, end, 1)
        moves = moves + move(tmp, start, end, num - 1)
        return moves
```

This is Towers of Hanoi. Every time we add one disc, we double the number of moves. That's exponential time, or $O(2^n)$.

Be Careful of Built-in Runtimes!

```
def countAll(lst):
    for i in range(len(lst)):
        count = lst.count(i)
        print(i, "occurs", count, "times")
```

This is actually $O(n^2)$, because each call to `lst.count(i)` takes O(n) time.

We'll let you know on assignments and exams when a built-in operation is not constant time.

Activity: Calculate the Big-O of Code

Activity: predict the Big-O runtime of the following piece of code.

```
def sumEvens(lst): # n = len(lst)
    result = 0
    for i in range(len(lst)):
        if lst[i] % 2 == 0:
            result = result + lst[i]
    return result
```

Submit your answer to Piazza when you're done.

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