UNIT 14A
The Limits of Computing: Intractability

Announcement

- If you made a special arrangement for the final exam with me and have not gotten an email from me, come and see me at the end of the lecture.

Last Week

- Last programming assignment PA11 is due Friday, last day of classes
- Lab Exam 2 during the recitations
  - A sample exam posted on the Schedule page
  - More exam samples on the Resources page
  - Expanded Ruby Drills on the Resources page
- OLI Module 6: Computability . Covers the material of our last unit

Computability

- Can a computer solve any possible problem that we pose to it as a program?
- In this unit we will learn that
  - Some problems are intractable: solvable but requires so much time (or space) that effectively out of reach
  - Some problems are unsolvable: no matter how fast the computer is (how big the memory is) it is impossible to solve them

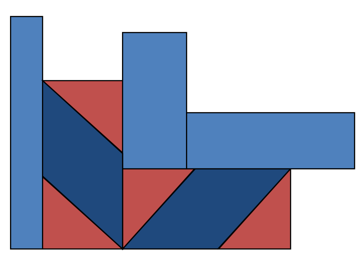
Why Study Unsolvability?

- Practical: If we know that a problem is unsolvable we know that we need to simplify or modify the problem
- Cultural: Gain perspective on computation

Decision Problems

- A specific set of computations are classified as decision problems.
- An algorithm describes a **decision problem** if its output is simply YES or NO, depending on whether a certain property holds for its input.
- Example:

  Given a set of N shapes, can these shapes be arranged into a rectangle?



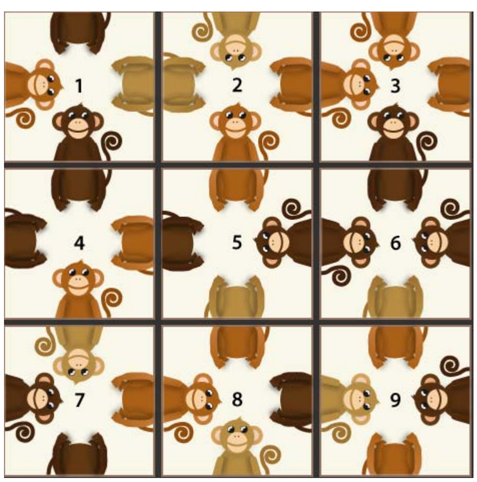
The Monkey Puzzle

- Given:
  - A set of N square cards whose sides are imprinted with the upper and lower halves of colored monkeys.
  - N is a square number, such that $N = M^2$.
  - Cards cannot be rotated.

decision problem

- Problem:
  - Determine if an arrangement of the N cards in an M X M grid exists such that each adjacent pair of cards display the upper and lower half of a monkey of the same color.

Example



- Is there a YES answer to the decision problem?
- If there is, is the problem tractable in general?

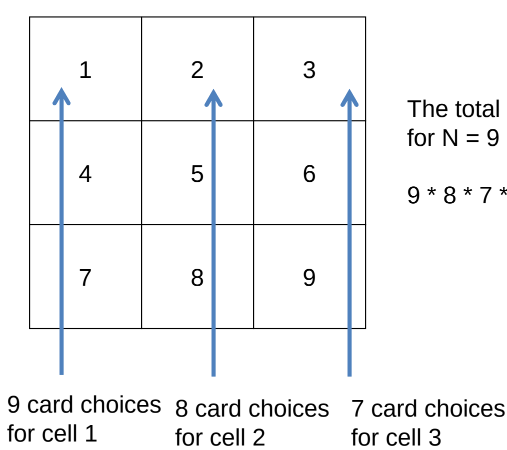
Algorithm

Simple brute-force algorithm:

- Pick one card for each cell of M X M grid.
- Verify if each pair of touching edges make a full monkey of the same color.
- If not, try another arrangement until a solution is found or all possible arrangements are checked.
- Answer "YES" if a solution is found. Otherwise, answer "NO" if all arrangements are analyzed and no solution is found.

Analysis

Suppose there are N = 9 cards (M = 3)



The total number of unique arrangements for N = 9 cards is:

9 * 8 * 7 * .... *1 = 9! (9 factorial)

goes on like this

Analysis (cont'd)

For N cards, the number of arrangements to examine is N! (N factorial)

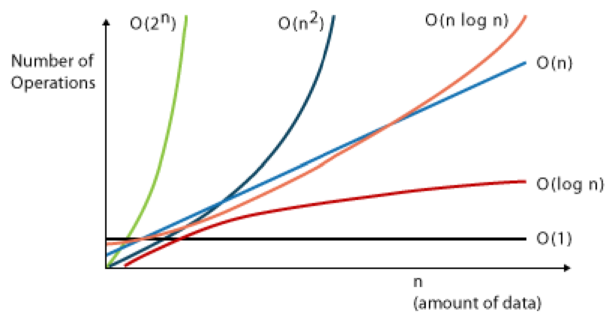
If we can analyze one arrangement in a microsecond:

N	Time to analyze all arrangements
9	362,880 μs
16	20,922,789,888,000 μs (app. 242 days)
25	15,511,210,043,330,985,984,000,000 μs

Reviewing the Big O Notation (1)

- We use the big O notation to indicate the relationship between the amount of data to be processed and the corresponding amount of work.
- For the Monkey Puzzle
  - Amount of data to be processed: the number of board arrangements
  - Amount of work: Number of operations to check if the arrangement solves the problem
- For very large n (size of input data), we express the number of operations as the (time) order of complexity.

Growth of Some Functions



Big O notation:

gives an asymptotic upper bound

ignores constants

Any function *f(n)* such that $f(n) \le c\, n^2$ for large *n* has $O(n^2)$ complexity

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Quiz on Big O

- What is the complexity in big O for the following descriptions
  - The amount of computation does not depend on the size of input data O(1)

    For example, work is always 3 operations, or 5 operations
  - If we double the input size the work is doubles, if we triple it the work is 3 times as much O(n)

    For example, work is 2n + 5, or 8n
  - If we double the input size the work is 4 times as much, if we triple it the work is 9 times as much $O(n^2)$

    For example, work is $2n^2 + 5$, or $8n^2$
  - If we double the input size, the work has 1 additional operation

    O(log n) For example, work is 2 lg n + 5

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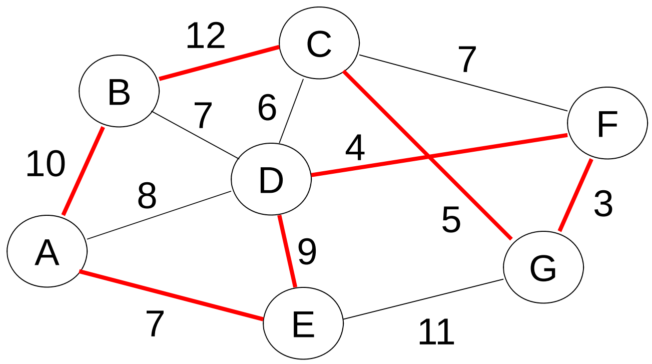
Classifications

- Algorithms that are *$O(N^k)$* for some fixed k are polynomial-time algorithms.
  - *$O(1)$, $O(\log N)$, $O(N)$, $O(N \log N)$, $O(N^2)$*
  - reasonable, tractable
- All other algorithms are super-polynomial-time algorithms.
  - *$O(2^N)$, $O(N^N)$, $O(N!)$*
  - unreasonable, intractable

Traveling Salesperson

- Given: a weighted graph of nodes representing cities and edges representing flight paths (weights represent cost)
- Is there a route that takes the salesperson through every city and back to the starting city with cost no more than K?
  - The salesperson can visit a city only once (except for the start and end of the trip).

Traveling Salesperson

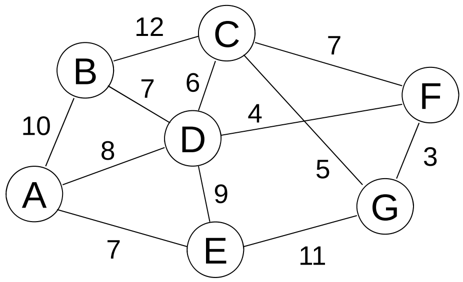


Is there a route with cost at most 52? YES (Route above costs 50.)
Is there a route with cost at most 48? YES? NO?

Analysis

- If there are N cities, what is the maximum number of routes that we might need to compute?
- Worst-case: There is a flight available between every pair of cities.
- Compute cost of every possible route.
  – Pick a starting city
  – Pick the next city (N-1 choices remaining)
  – Pick the next city (N-2 choices remaining)
  – ...
- Maximum number of routes: __________

Number of Paths to Consider



Number of all possible paths = Number of all possible permutations of N nodes = N!

Observe ABCGFDE is equivalent to BCGFDEA

Number of all possible unique paths = N! / N = N – 1!

Observe ABCGFDE has the same cost as EDFGCBA

Number of all possible paths to consider = (N – 1)! / 2

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Analysis

- If there are N cities, what is the maximum number of routes that we might need to compute?
- Worst-case: There is a flight available between every pair of cities.
- Compute cost of every possible route.
  - Pick a starting city
  - Pick the next city (N-1 choices remaining)
  - Pick the next city (N-2 choices remaining)
  - ...
- Worst-case complexity: O(N!)

Note: $N! > 2^N$ for every N > 3.

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Map Coloring

- Given a map of N territories, can the map be colored using K colors such that no two adjacent territories are colored with the same color?
- K = 4: Answer is always yes.
- K = 2: Only if the map contains no point that is the junction of an odd number of territories.

Map Coloring

- Given a map of N territories, can the map be colored using **3** colors such that no two adjacent territories are colored with the same color?

Analysis

- Given a map of N territories, can the map be colored using **3** colors such that no two adjacent territories are colored with the the same color?
  - Pick a color for territory 1 (3 choices)
  - Pick a color for territory 2 (3 choices)
  - ...
- There are $3 * 3 * \ldots *3 = 3^N$ possible colorings.

Satisfiability

- Given a Boolean formula with N variables using the operators AND, OR and NOT:
  - Is there an assignment of boolean values for the variables so that the formula is true (satisfied)?
    Example: (X AND Y) OR (NOT Z AND X)
  - Truth assignment: X = True, Y = True, Z = False.
- How many assignments do we need to check for N variables?
  - Each symbol has 2 possibilities ... 2^N assignments

The Big Picture

- Intractable problems are solvable if the amount of data (*N*) that we're processing is small.
- But if *N* is not small, then the amount of computation grows exponentially and the solutions quickly become intractable (i.e. out of our reach).
- Computers can solve these problems if *N* is not small, but it will take far too long for the result to be generated.
  – We would be long dead before the result is computed.

What's Next

- For a specific decision problem, is there single tractable (polynomial-time) algorithm to solve any instance of this problem?
- If one existed, can we use it to solve other decision problems?
- What is one of the big computational questions to be answered in the 21st century?