UNIT 12B
Continuous-Time Simulations

Announcements

- Nominate your CAs for a teaching award by sending email to gkesden@gmail.com by 11:59pm on April 19.
- If you have final exam conflicts or need a special accommodation email me dilsun@cs.cmu.edu with the dates you can make the exam other than the scheduled date of May 13
- We sent you email informing you about your current grade in the course!

Why Do Simulations?

- To predict the behavior of a system.
  - Will this building survive an earthquake?
- To test a theory against data.
  - Do the predictions generated by these equations match what we observe in the real world?
- To explore consequences of assumptions.
  - How quickly does the flu virus spread?

3

Continuous-Time Simulations

- Often used to model physical phenomena involving forces acting on objects.
- Is “time” really continuous?
  - Philosophical question. No one knows.
  - Just pretend it is.
- Is simulated time continuous?
  - No. It’s divided into discrete time steps.
  - But they can be as small as we like.

4

The Solar System

- Newton’s theory: Planets and other bodies move according to the gravitational effects of the objects around them
- But, there is no equation to solve to determine the precise locations at any point in the future for $3 \leq N$ bodies.
- N-body problem: Describing the motion of a collection of bodies

5

N-Body Simulations

- With just two bodies, we can write a simple formula to calculate their positions at any future time, given their starting positions.
- But with 3 or more bodies, no formula exists for this, because the system is highly nonlinear, and potentially chaotic.
- Our only recourse is simulation.

6

This Lecture

- Using simulation to predict future locations of bodies
  - Astronomers use simulations to predict locations of satellites, plan space travel, track dangerous asteroids etc.
- Main idea of the simulation: Start with the current location and heading of each planet. Then repeatedly
  - Determine where the planets would be a short time later if they move according to a straight line
  - Calculate adjustments to headings

7

Simulating Gravitational Attraction

Newton's law of universal gravitation:

$$F = G \cdot m_1 \cdot m_2 / d^2$$

where G = gravitational constant,
m_1 and m_2 are the masses, and
d is the distance between them.

8

Force and Acceleration

- Newton's second law: if some external force is applied to a body then the body accelerates (its velocity changes)

$$F = ma$$

9

Moving A Single Body

- Calculate the force and acceleration influencing the body at a given time
- Suppose that acceleration is constant for a given interval of time and calculate the velocity and distance moved

10

Example: 2-body system

- Simulate the motion of a body as a result of gravitational force applied by the Earth.
- This example offers a simple instance of our N-body problem.
- For N = 2, there is an equation that predicts how far the body can go in a given amount of time. This lets us check accuracy of our simulation.

Example: 2-body system (cont'd)

- An object near the surface of the Earth is pulled toward the center of the Earth by the force of gravity.
- Distance = ½ g × t^2
  - g = 9.8 m/ s^2 is gravitational acceleration
  - Equivalent to saying t = sqrt(2 ×d/g)
- If you drop a stationary melon, the distance it travels in 1 seconds is ½ 9.8 x 1^2 = 4.9, ignoring friction etc.

2-Body Example in RubyLabs

```
>> include SphereLab
>> b = make_system(:melon)
⇒[melon: …, earth: …]
```

Every object has a mass, position, and velocity.

```
>> view_melon(b)
>> position_melon(b, 50)
⇒50
>> update_melon(b, 0.5)
=>47.54
```

The output shows the distance to earth surface.

13

Checking Accuracy

```
>> update_melon(b, 1.0)
⇒90.18
```

The melon fell 9.81 meters but the equation from the previous slide predicts 4.9 meters. Where does this error come from?

14

Time-step vs. Accuracy

Instead of update_melon(b,1.0), if we used update(b,0.1) to times, would the error be smaller or bigger?

>> 10.times {update_melon(b, 0.1) ; b[0].height}

$\Rightarrow$94.6018…

The smaller the time-step the more accurate our simulation.

15

Integrating Acceleration

- When an object accelerates, its velocity v(t) changes. How can we model this?
- Divide time into tiny steps Δt.
- Re-calculate the velocity at each time step.

  $v(t+\Delta t) = v(t) + a(t) \cdot \Delta t$
- Smaller Δt brings greater accuracy.

16

Velocity Is Rate of Change of Position

- If an object has non-zero velocity, its position is changing.
- We can use the same integration trick to update the body's position based on velocity.

  $x(t+\Delta t) = x(t) + v(t) \cdot \Delta t$

17

Force Vectors

An object can be moving in multiple dimensions at once. Try the example below:

```
>> view_melon(b)
>> position_melon(b, 100)
=>100
>>b[0].velocity.x = 5
```

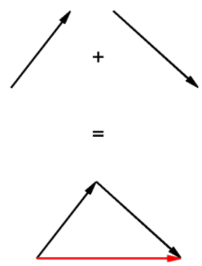
Sets velocity of melon in the horizontal right direction to 5.

```
>> drop_melon(b, 0.1)
=>4.5
```

18

Force Vectors in RubyLabs

- SphereLab bodies use vectors to keep track of positions, velocities, and accelerations: (x, y, z) coordinates
- Forces are additive and vector addition is like ordinary addition



This vectors in this example has 0 for the z coordinate.

The north and south vectors cancel out each other

The east vectors add up

19

Moving A Single Body

- Calculate all the force vectors influencing the body
- Add the vectors together to determine the cumulative force

20

Falling Body Experiment

>> b = make_system(:fdemo)

The first body is supposed to fall toward the other 5 bodies that are stationary.

>> update_one(b[0], b[1..5], 1.0)

Simulates the motion of the body for a time step of 1.0.

What happens when we repeat the experiments 10s of times?

21

Chaotic Systems

Experiment with slight changes in initial conditions

>>b=make_system(:fdemo)

>>b[0].position

⇒(81.471, 145.85) *# initial position of b[0]*

>>b[0].position.x += 1

⇒(82.471, 145.85)

What difference do you observe in the trajectory of b[0]?

22

Moving Multiple Bodies

- At each time step move each body by calculating the force vectors in each direction
- Suppose we are given a method interaction(a,b) that calculates the gravitational force between the bodies a and b
- We need to compute all pairwise interactions.
- How many calculations are there?
  - N + N-1 + N-2 + … = N × N+1/2 => $O(N^2)$

23

Simulating The Solar System

```
>> include SphereLab
>> b = make_system(:solarsystem)
>> view_system(b[0..4], :dash => 1)
>> 365.times {
       update_system(b, 86459); sleep(0.1) }
```

Notice that the orbits are elliptical (Kepler).

24

Simulation At Extreme Scales

- Cosmologists use simulations to study the formation of galaxies (clusters of stars), and even clusters of galaxies.
- At the other extreme, physicists simulate individual atoms and molecules, e.g., to model chemical reactions.

25