

The Naïve Bayes Algorithm

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Bayes Rule

Bayes Rule:
$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

$P(A)$ prior

$P(A|B)$ posterior

...by no means merely a curious speculation in the doctrine of chances, but necessary to be solved in order to a sure foundation for all our reasonings concerning past facts, and what is likely to be hereafter.... necessary to be considered by any that would give a clear account of the strength of *analogical* or *inductive reasoning*...



Bayes, Thomas (1763) An essay towards solving a problem in the doctrine of chances. *Philosophical Transactions of the Royal Society of London*, 53:370-418

Applying Bayes Rule

Bayes Rule:

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

$$P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|\bar{A})P(\bar{A})}$$

A = you got flu B = you just coughed

$P(A) = 0.05$, $P(B|A) = 0.8$, $P(B|\bar{A}) = 0.2$

What is $P(\text{flu} | \text{cough}) = P(A | B)$?

Learning and the Joint Distribution

Suppose we want to learn the function $f: \langle C, H \rangle \rightarrow W$

Equivalently, $P(W \mid C, H)$

One solution: learn joint distribution from data, calculate $P(W \mid C, H)$

College Degree	Hours worked	Wealth	prob
No	40.5-	Medium	0.253122
No	40.5-	Rich	0.0245895
No	40.5+	Medium	0.0421768
No	40.5+	Rich	0.0116293
Yes	40.5-	Medium	0.331313
Yes	40.5-	Rich	0.0971295
Yes	40.5+	Medium	0.134106
Yes	40.5+	Rich	0.105933

$$\text{e.g., } P(W = \text{rich} \mid C = \text{no}, H = 40.5 -) = \frac{0.0245895}{0.0245895 + 0.253122}$$

Idea: learn classifiers by learning $P(Y | X)$

Consider $Y = \text{Wealth}$

$X = \langle \text{CollegeDegree}, \text{HoursWorked} \rangle$

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Yes	40.5+	Rich	0.105933

College Degree	Hours worked	$P(\text{rich} C,HW)$	$P(\text{medium} C,HW)$
No	< 40.5	.09	.91
No	> 40.5	.21	.79
Yes	< 40.5	.23	.77
Yes	> 40.5	.38	.62

One approach: use this representation to learn $P(Y|X)$.

Are we done?!?

One approach: use this representation to learn $P(Y|X)$.

Main problem: learning $P(Y|X)$ might require more data than we have...

Example:

Consider learning joint distributions with 100 attributes

Number of rows in this table? $2^{100} \sim 100^{10} \sim 10^{30}$

Number of people on Earth? 10^9

Fraction of rows with 0 training examples: 0.9999

What to do?

1. Be smart about how to estimate probabilities
2. Be smart about how to represent joint distributions

Be smart about how to estimate probabilities

Principle 1: Maximum Likelihood Estimation

Choose parameter $\hat{\theta}$ that maximizes likelihood of observed data $P(\text{data}|\hat{\theta})$

$$\hat{\theta}_{\text{MLE}} = \frac{\alpha_H}{\alpha_T + \alpha_H}$$

Principle 2: Maximum A posteriori Probability

Choose parameter $\hat{\theta}$ that maximizes likelihood the posterior prob $P(\hat{\theta}|\text{data})$

$$\hat{\theta}_{\text{MAP}} = \frac{\alpha_H + \text{\#halucinated_Hs}}{(\alpha_T + \text{\#halucinated_Ts}) + (\alpha_H + \text{\#halucinated_Hs})}$$

Be smart about how to represent joint distributions

Naïve Bayes algorithms assumes that

$$P(X_1, X_2, \dots, X_n | Y) = \prod_i P(X_i | Y)$$

i.e., X_i and X_j are conditionally independent given Y , for all $i \neq j$

Conditional Independence

Definition

X is **conditionally independent** of **Y** given **Z** iff

the probability distribution governing X is independent of Y, given the value of Z.

$$(\forall x, y, z): \quad P(X = x | Y = y, Z = z) = P(X = x | Z = z)$$

We often write as $P(X|Y, Z) = P(X|Z)$

E.g., $P(\text{Thunder} | \text{Rain}, \text{Lightening}) = P(\text{Thunder} | \text{Lightening})$

Note: does NOT **mean** that Thunder is independent of Rain.

Conditional Independence

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E.g., 3 Boolean random variables to describe the weather: Thunder, Rain, Lightning.

$$P(\text{Thunder} | \text{Rain}, \text{Lightning}) = P(\text{Thunder} | \text{Lightning})$$

Thunder is independent of Rain given Lightning. Lightning causes Thunder, once we know whether or not there is Lightning, no additional information about Thunder is provided by the value of Rain.

It does NOT mean that Thunder is independent of Rain.

Clear dependence of Thunder on Rain in general, but there is no conditional dependence once we know the value of Lightning.

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Equivalent to $P(X, Y|Z) = P(X|Z)P(Y|Z)$

Conditional Independence

Claim

X is conditionally independent of Y given Z iff $P(X|Y, Z) = P(X|Z)$

Equivalent to $P(X, Y|Z) = P(X|Z)P(Y|Z)$

$$\begin{aligned} P(X, Y|Z) &= P(X|Y, Z)P(Y|Z) \\ &= P(X|Z)P(Y|Z) \end{aligned}$$

Conditional Independence

Claim

If X_i and X_j are conditionally independent given Y , for all $i \neq j$

$$P(X_1, X_2, \dots, X_n | Y) = \prod_i P(X_i | Y)$$

If X_1, \dots, X_n, Y are all Boolean, how many parameters do we need to describe $P(X_1, X_2, \dots, X_n | Y)$ and $P(Y)$?

- Without the conditional independence assumption: $2(2^n - 1) + 1$
- With conditional independence assumption: $2n + 1$

Naïve Bayes in a Nutshell

Bayes Rule: $P(Y = y_k | X_1, \dots, X_n) = \frac{P(Y = y_k) P(X_1, \dots, X_n | Y = y_k)}{P(X)}$

If X_i and X_j are conditionally independent given Y , for all $i \neq j$

$$P(Y = y_k | X_1, \dots, X_n) = \frac{P(Y = y_k) \prod_i P(X_i | Y = y_k)}{P(X)}$$

So, to pick the most probably Y for $X^{\text{new}} = (X_1^{\text{new}}, X_2^{\text{new}}, \dots, X_n^{\text{new}})$

$$Y^{\text{new}} = \operatorname{argmax}_{y_k} P(Y = y_k) \prod_i P(X_i^{\text{new}} | Y = y_k)$$

Naïve Bayes: discrete X_i

Training phase (input: training examples)

- For each value y_k , estimate $\pi_k = P(Y = y_k)$; get $\widehat{\pi}_k$
- For each value x_{ij} of attribute X_i estimate $\theta_{i,j,k} = P(X_i = x_{ij} | Y = y_k)$; get $\widehat{\theta}_{i,j,k}$

Testing phase:

- Classify $X^{\text{new}} = (X_1^{\text{new}}, X_2^{\text{new}}, \dots, X_n^{\text{new}})$

$$Y^{\text{new}} = \operatorname{argmax}_{y_k} \widehat{\pi}_k \prod_i \widehat{\theta}_{i,\text{new},k}$$

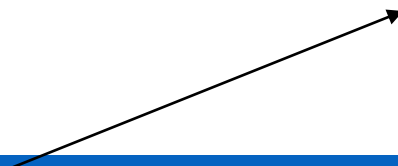
[Ideal rule: $Y^{\text{new}} = \operatorname{argmax}_{y_k} P(Y = y_k) \prod_i P(X_i^{\text{new}} | Y = y_k)$]

Estimating parameters Y, X_i discrete

Maximum Likelihood Estimation

- For each value y_k , get $\hat{\pi}_k = \hat{P}(Y = y_k) = \frac{\#D(Y=y_k)}{|D|}$
- For each value x_{ij} of attribute X_i estimate $\theta_{i,j,k} = P(X_i = x_{ij} | Y = y_k)$;

$$\text{get } \hat{\theta}_{i,j,k} = \hat{P}(X_i = x_{ij} | Y = y_k) = \frac{\#D(X_i=x_{ij} \wedge Y=y_k)}{\#D(Y=y_k)}$$



Number of items in
dataset D for which $Y=y_k$

Subtlety 1: Violation of the Naïve Bayes Assumption

- Usually features are not conditionally independent given the label

$$P(X_1, X_2, \dots, X_n|Y) \neq \prod_i P(X_i|Y)$$

- Nonetheless, NB is widely used:
 - NB often performs well, even when assumption is violated
 - [Domingos & Pazzani '96] discuss some conditions for good performance

Subtlety 2: Insufficient Training Data

- What if we never see a training instance where $X_1 = a$ and $Y = b$?

e.g., $Y = \text{SpamEmail}$, $X = \text{"Earn"}$

$$\hat{P}(X_1 = a|Y = b) = 0 \qquad \hat{P}(X_1 = a|Y = b) = 0$$

- Thus no matter what the values X_2, \dots, X_n take, we obtain:

$$\hat{P}(Y = b|X_1 = a, X_2, \dots, X_n) = 0$$

$$\hat{P}(X_1 = a, X_2, \dots, X_n|Y) = \hat{P}(X_1 = a) \prod_{i \neq 1} P(X_i|Y)$$

- Solution: use MAP estimate!!!!

Estimating parameters Y, X_i discrete

Maximum Likelihood Estimation

- For each value y_k , get $\hat{\pi}_k = \hat{P}(Y = y_k) = \frac{\#D(Y=y_k)+1}{|D|+1K}$
- For each value x_{ij} of attribute X_i estimate $\theta_{i,j,k} = P(X_i = x_{ij} | Y = y_k)$;

$$\text{get } \hat{\theta}_{i,j,k} = \hat{P}(X_i = x_{ij} | Y = y_k) = \frac{\#D(X_i=x_{ij} \wedge Y=y_k)+1}{\#D(Y=y_k)+1J}$$

J - number of distinct values that feature i can take; 1 determines the strength of this smoothing; assume the hallucinated examples are spread evenly over the possible values of X_i ; so, number of hallucinated examples is $1J$.

What you should know...

- Naïve Bayes classifier
 - What's the assumption
 - Why we use it
 - How do we learn it
 - Why is Bayesian estimation important