Policy gradient theorem

d = number of parameters in policy, so $\theta \in \mathbb{R}^d$

$$J(\theta) = \mathbb{E}_{\pi_\theta}[r_1 + r_2 + \ldots + r_T]$$

- On iteration m, get trajectory $\tau^m = (s_1^m, a_1^m, r_1^m \ldots, s_T^m, a_T^m, r_T^m)$ by following policy $\pi_\theta(a_t^m \mid s_t^m)$
- Define

$A_t^m = \sum_{i=t}^T r_i^m \in \mathbb{R}$ (empirical total reward starting from step t) — optionally subtract baseline estimate of $V^\pi(s)$

$u_t^m = \frac{d}{d\theta} \ln \pi_\theta(a_t^m \mid s_t^m) \in \mathbb{R}^d$ (*action score vector,* from autodiff)

$g^m = \sum_{t=1}^T A_t^m u_t^m \in \mathbb{R}^d$ (the gradient estimate)

Theorem:

g^m is an unbiased estimate of $\frac{d}{d\theta} J(\theta)$

even if we don't know anything about the environment

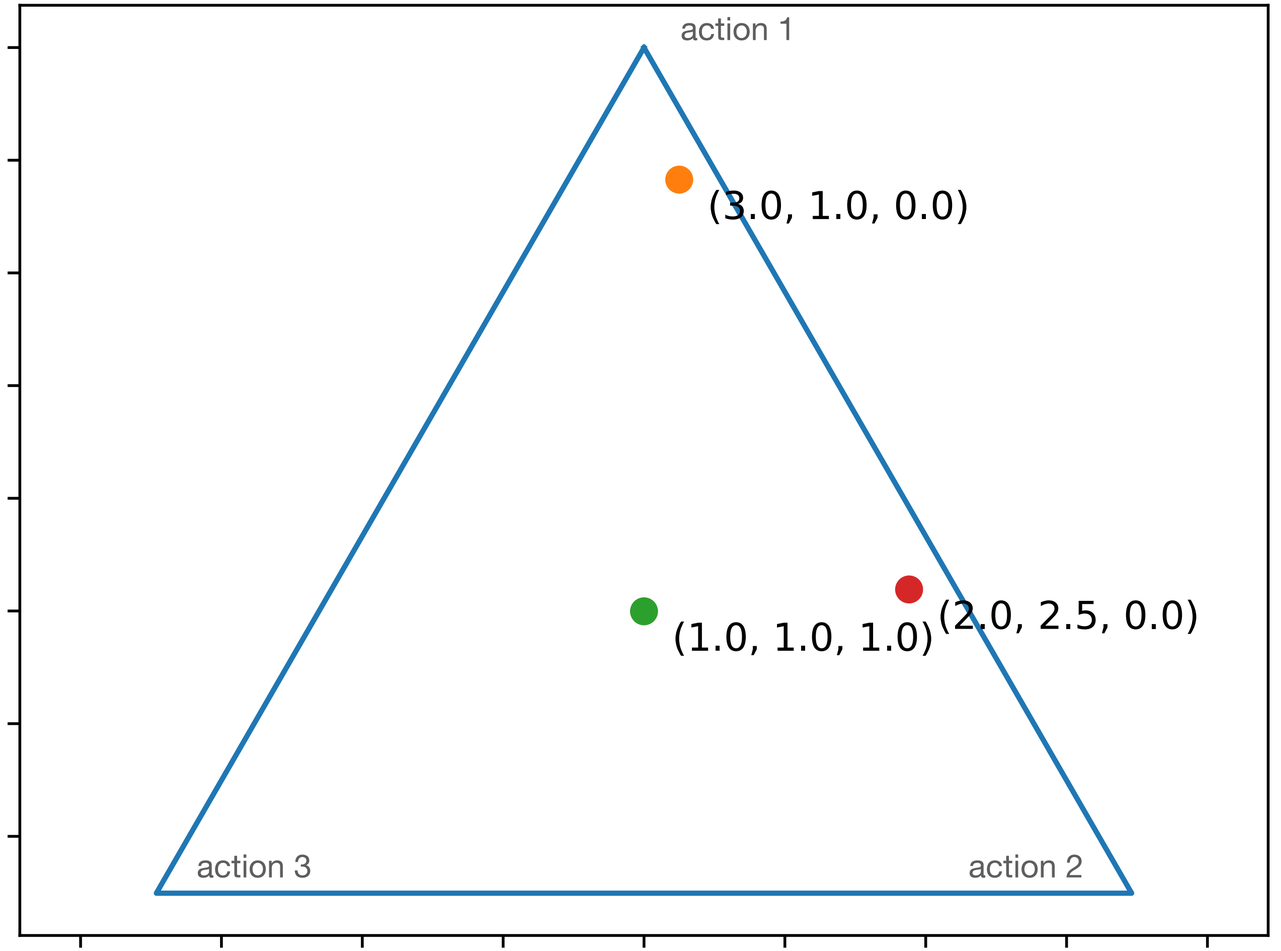
and ***even if*** environment is PO

Geoff Gordon

Softmax

3D action space, showing projection onto 2D simplex

$\text{sm}(z_1, z_2, z_3)$



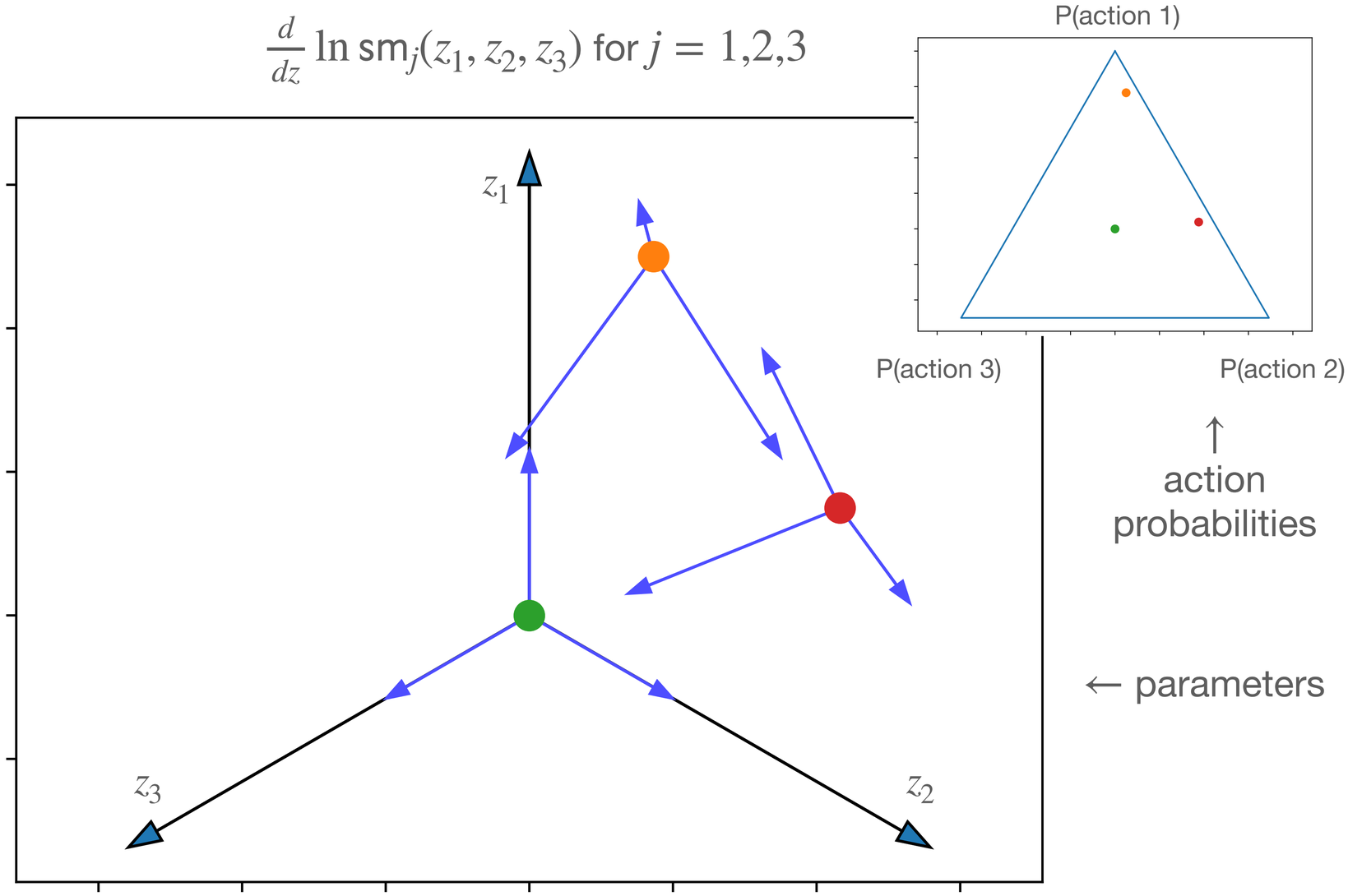
action
← probabilities

adding a constant doesn't change output:

$\text{sm}(z_1, z_2, z_3) = \text{sm}(z_1 + \text{const}, z_2 + \text{const}, z_3 + \text{const})$

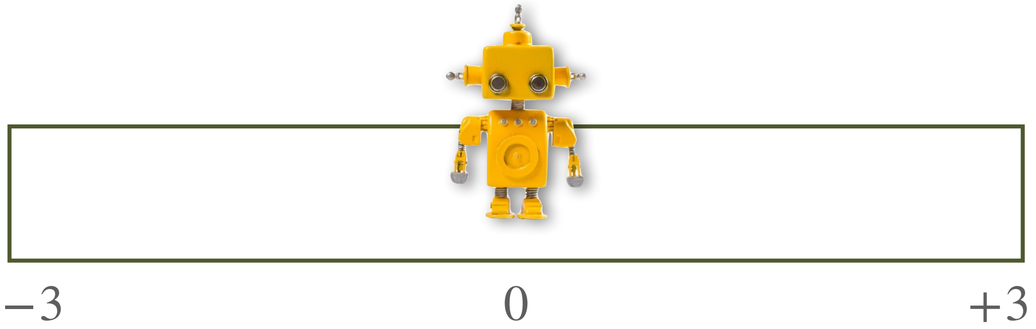
Gradient of log-softmax (action score vectors)

$\frac{d}{dz} \ln \mathrm{sm}_j(z_1, z_2, z_3)$ for $j = 1,2,3$



one arrow per action — measures how sensitive $\ln p(\text{act})$ is to parameters

REINFORCE gradient



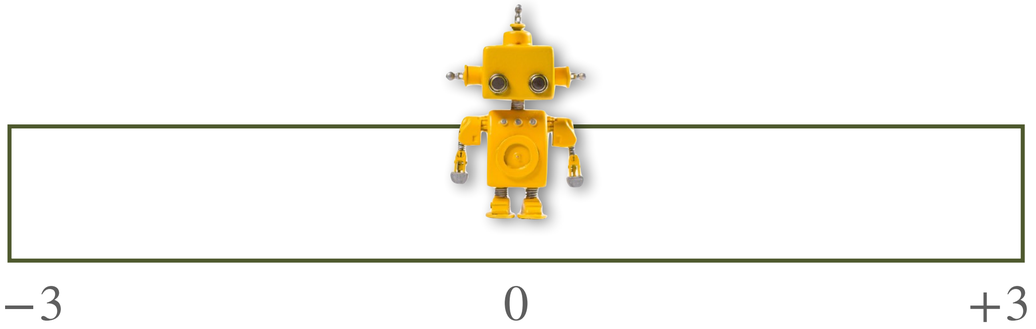
- Policy:

$$P(\mathsf{R} \mid x) = \frac{1}{1 + e^{-(wx+b)}}$$

$$P(\mathsf{L} \mid x) = \frac{1}{1 + e^{wx+b}}$$

$$\theta = (w, b)^\top$$

REINFORCE gradient



- Policy:

$$P(\mathsf{R} \mid x) = \frac{1}{1 + e^{-(wx+b)}}$$

$$P(\mathsf{L} \mid x) = \frac{1}{1 + e^{wx+b}}$$

$$\theta = (w, b)^\top$$

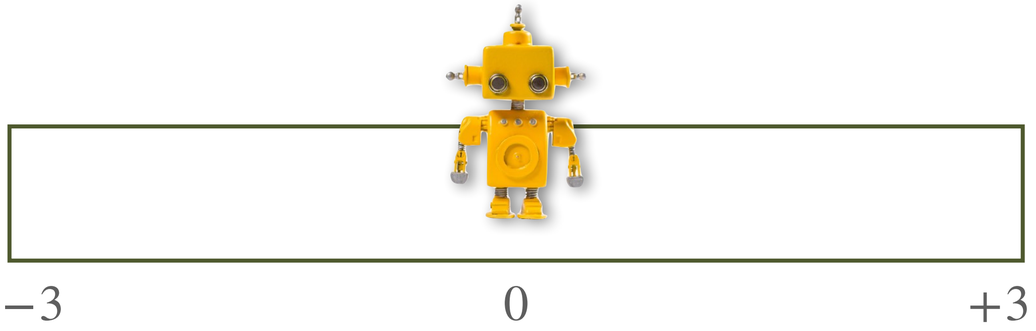
$$\nabla \ln P(\mathsf{R} \mid x) = P(\mathsf{L} \mid x) \begin{pmatrix} x \\ 1 \end{pmatrix}$$

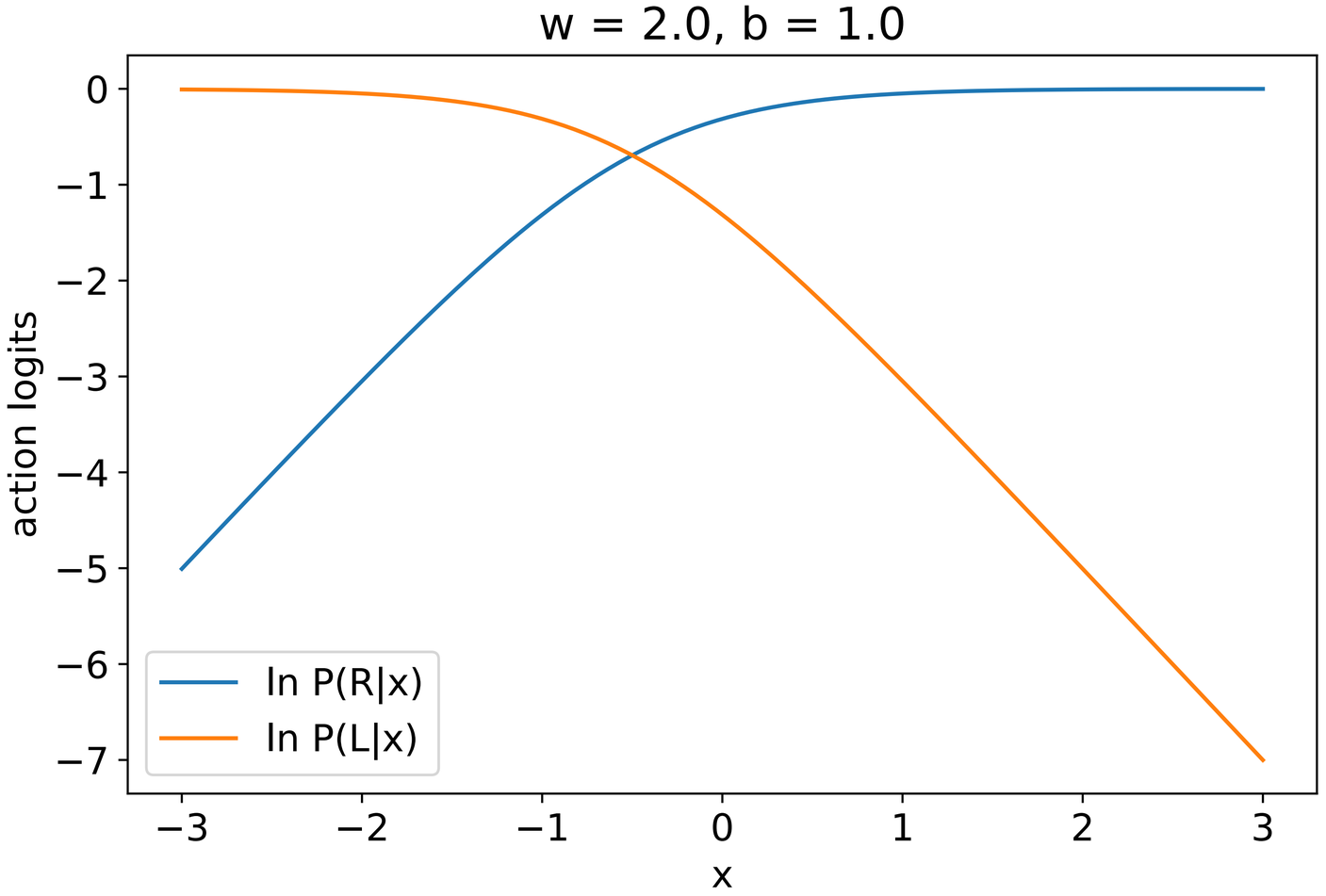
$$\nabla \ln P(\mathsf{L} \mid x) = P(\mathsf{R} \mid x) \begin{pmatrix} -x \\ -1 \end{pmatrix}$$

Action score example

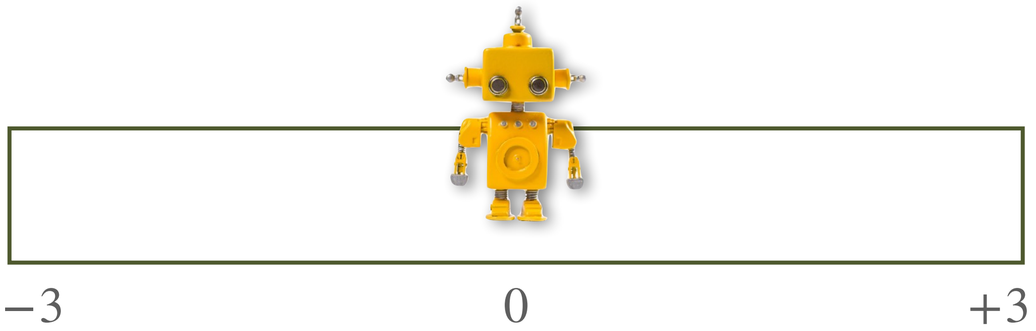
$$\nabla \ln P(\mathrm{L} \mid x = 2) =$$

$$P(\mathrm{R} \mid x = 2) \begin{pmatrix} -2 \\ -1 \end{pmatrix}$$

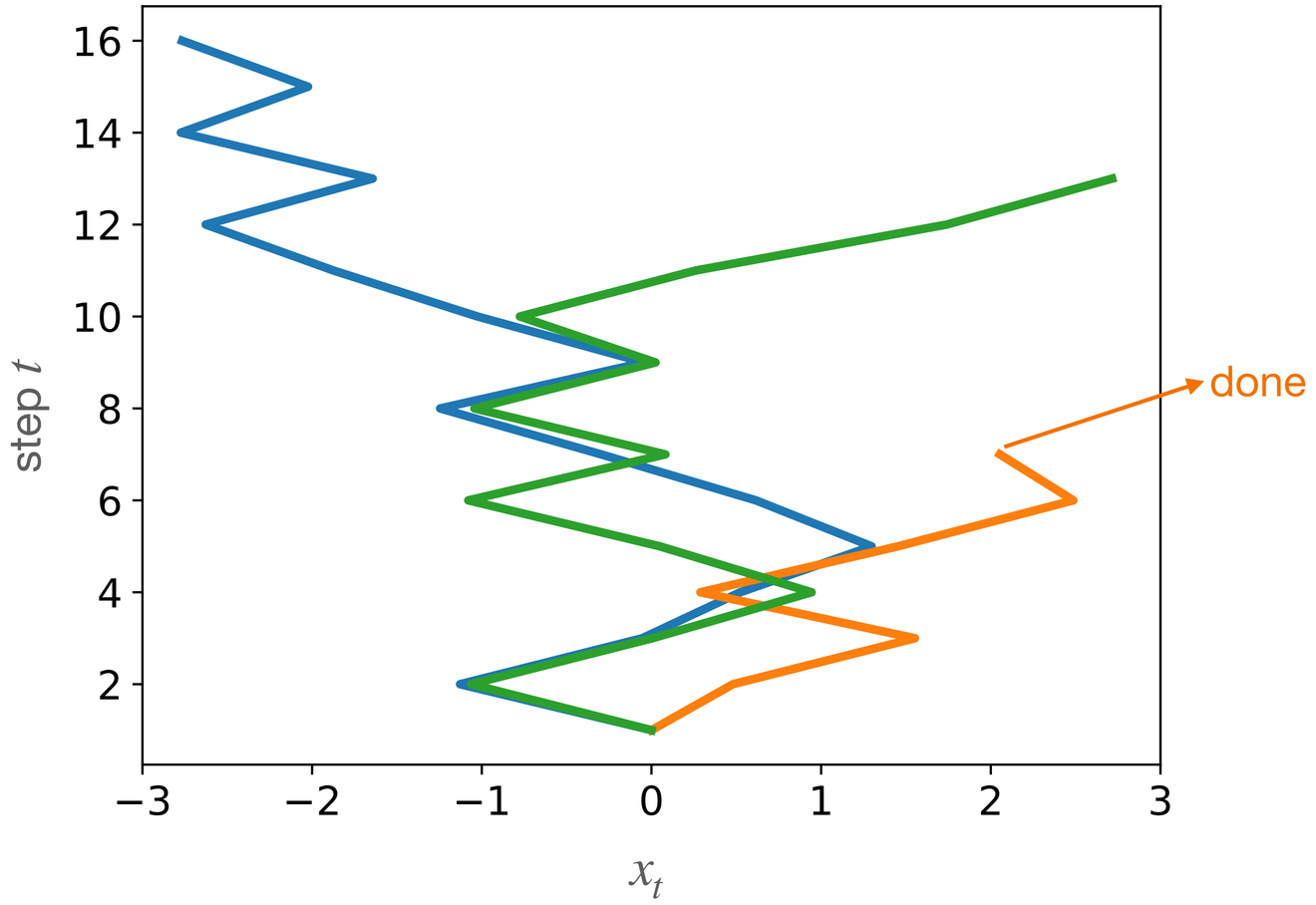




Collect data

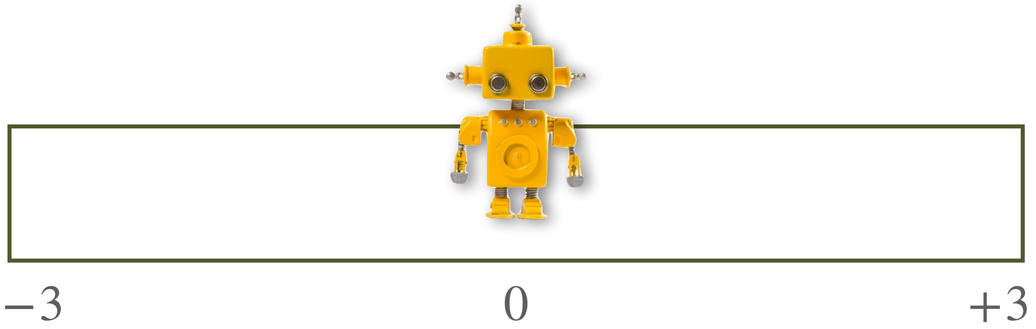


(x, a)	Q
(0, R)	−7
(.3, R)	−6
(1.6, L)	−5
(.2, R)	−4
(1.3, R)	−3
(2.4, L)	−2
(2.1, R)	−1



- Start at $w = b = 0$ (so π is uniform random at all x)

Calculate gradient estimate



(x, a)	Q
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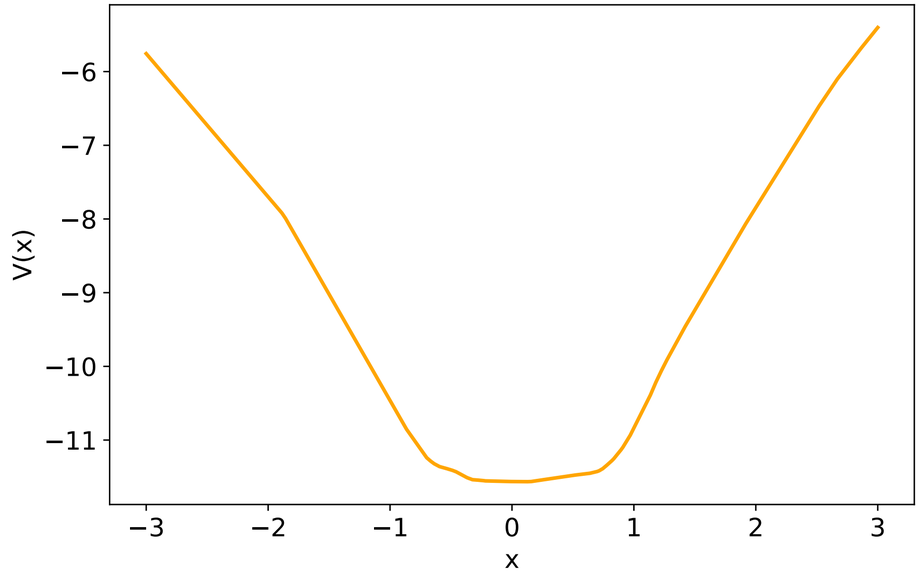
(1.3, R)	–3
(2.4, L)	–2
(2.1, R)	–1

$$g =$$



$$\nabla \ln P(\mathsf{R} \mid x) = P(\mathsf{L} \mid x) \begin{pmatrix} x \\ 1 \end{pmatrix}$$

$$\nabla \ln P(\mathsf{L} \mid x) = P(\mathsf{R} \mid x) \begin{pmatrix} -x \\ -1 \end{pmatrix}$$

$$g^m = \sum_{t=1}^{T} Q_t^m u_t^m$$

Geoff Gordon

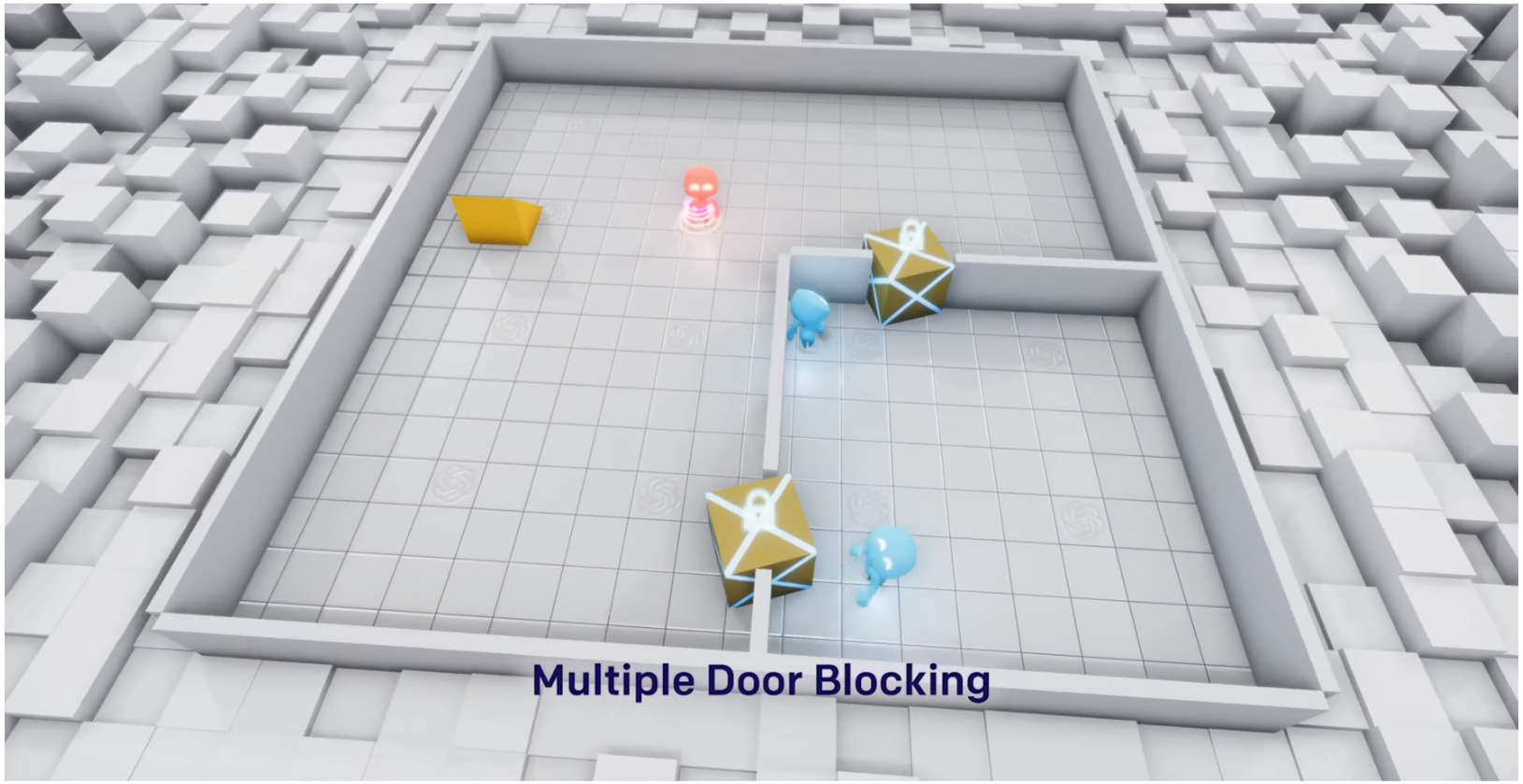
Summary, examples: policy gradients

- Most SOTA RL methods use gradient-based policy improvement
- Three main ways to estimate gradients:
  - Actor-critic
  - REINFORCE (uses observed returns only)
  - Reparameterization (+ critic derivative)
  - all based on variations of a similar derivation: avoid needing to know environment dynamics

key problem is to estimate advantages

- Some well-known examples:
  - AlphaGo used REINFORCE (baseline V = win probability estimated from a previous run)
  - PPO is partway between REINFORCE and actor-critic: mixes critic and actual returns w/ method called GAE
  - SAC uses actor-critic or reparameterization, depending on whether actions are discrete or continuous

A fun RL example



- OpenAI project using large-scale RL and evolutionary algorithms

Working with large models

10-701 Introduction to Machine Learning
Geoff Gordon and Pradeep Ravikumar

[with thanks to Matt Gormley]

Multi-stage training

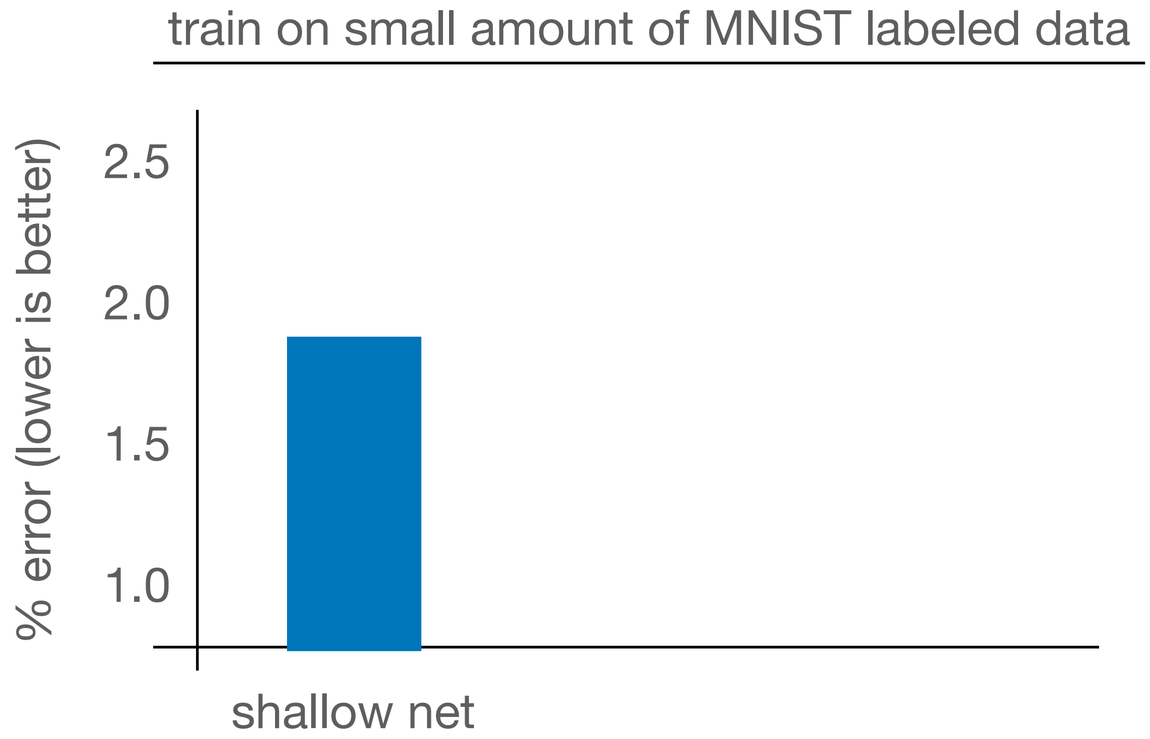
- Textbook ML: gather a dataset, define an objective function, run an optimizer, deliver a model
- At scale:
  - there might be many relevant sources of information (objective functions, datasets, …)
  - each with different advantages and disadvantages
  - none of which fully captures what we know about the problem
- Idea: combine!
  - e.g., gather several datasets, optimize a weighted sum of corresponding loss functions
  - or train in phases: optimize criterion A for a while, then switch to criterion B, then criterion C

for unknown reasons, training in phases seems to work better

[Bengio et al., 2006]

Multi-stage example

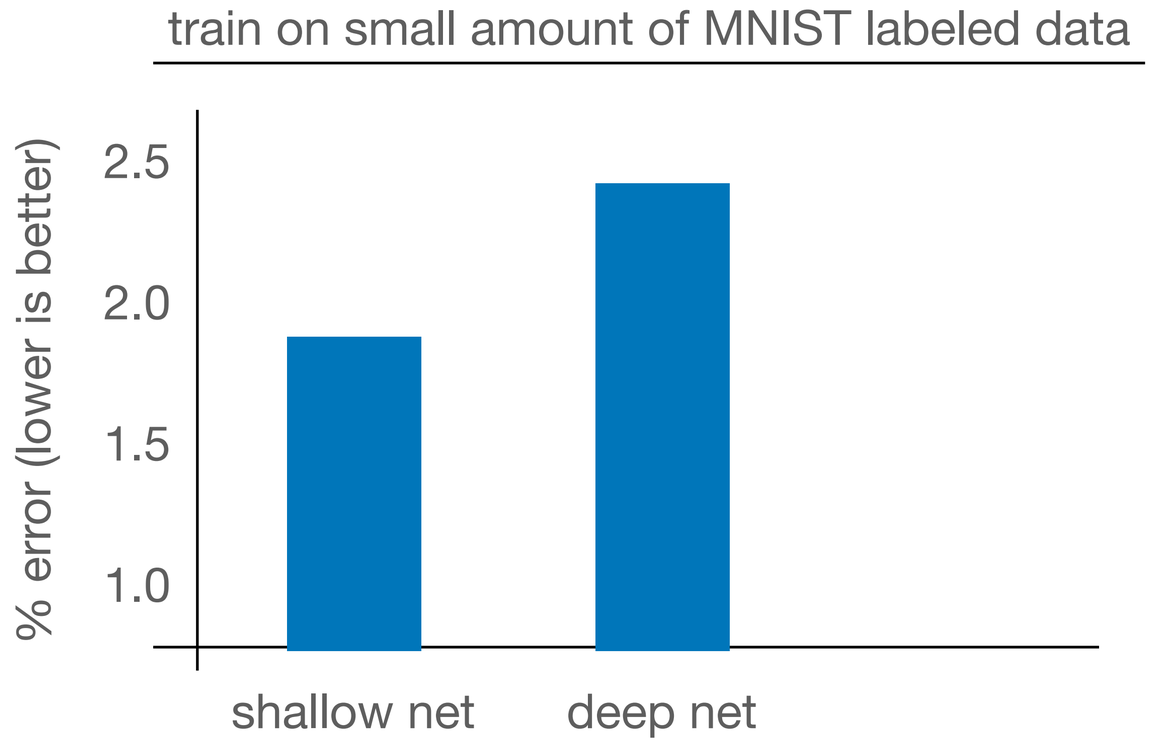
- MNIST digit classification



[Bengio et al., 2006]

Multi-stage example

- MNIST digit classification



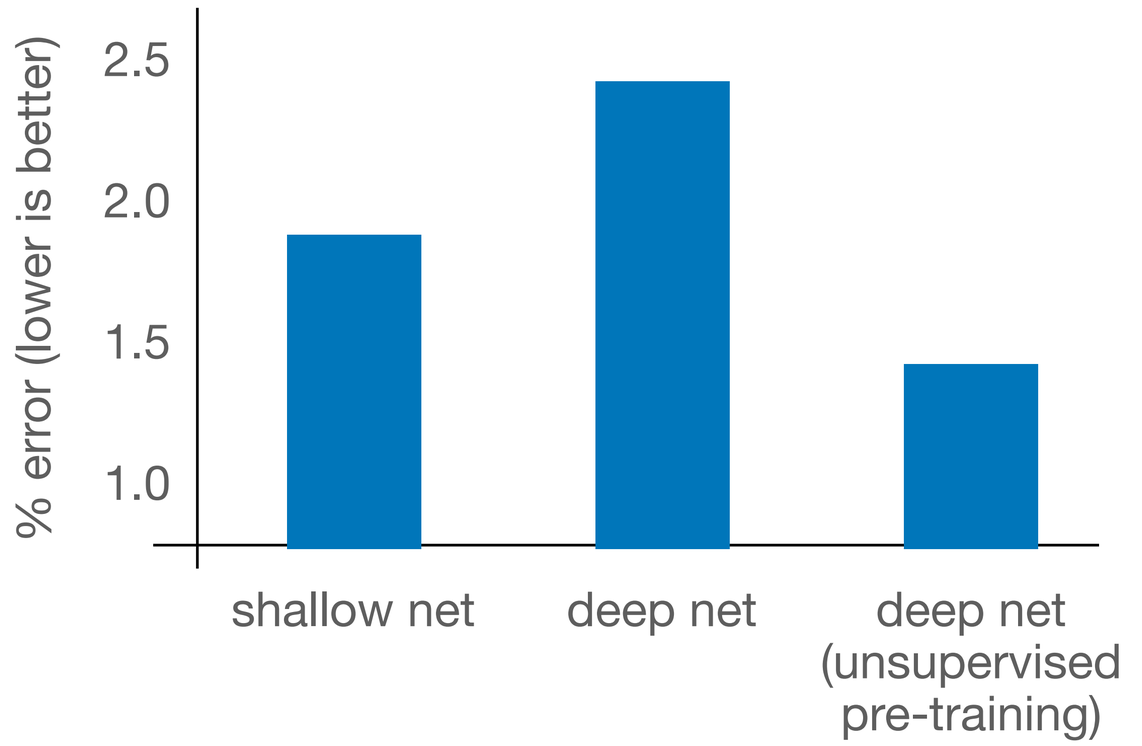
[Bengio et al., 2006]

Multi-stage example

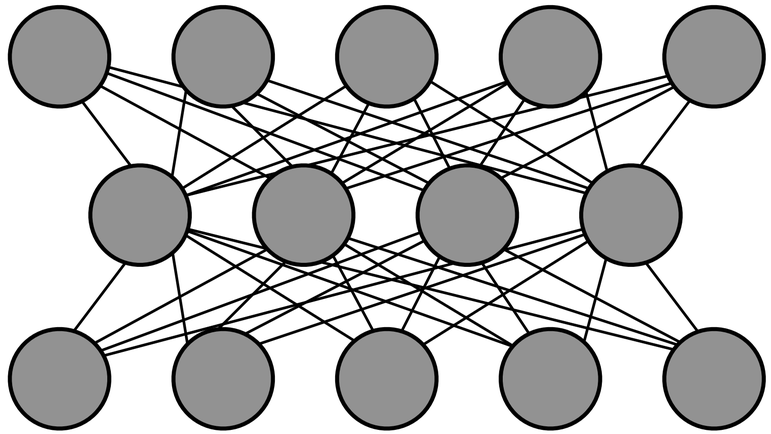
- MNIST digit classification

first train autoencoders on lots of unlabeled data

train on small amount of MNIST labeled data

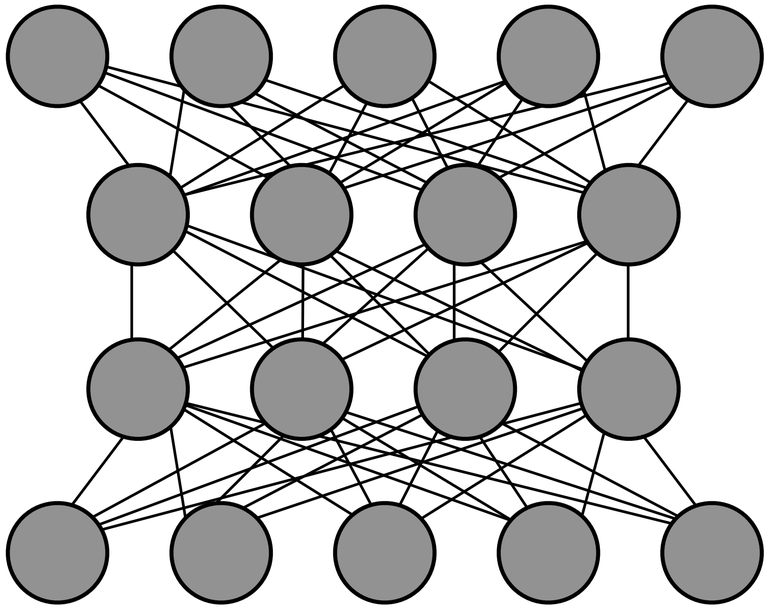


Autoencoder pre-training



- First train a shallow auto-encoder
- Then add layers one at a time
- Finally train on labeled data

Autoencoder pre-training



- First train a shallow auto-encoder
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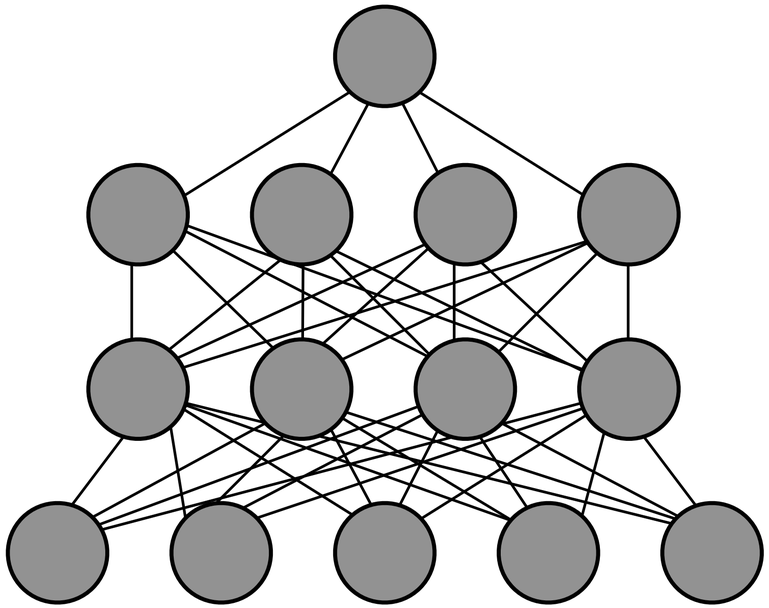
Autoencoder pre-training

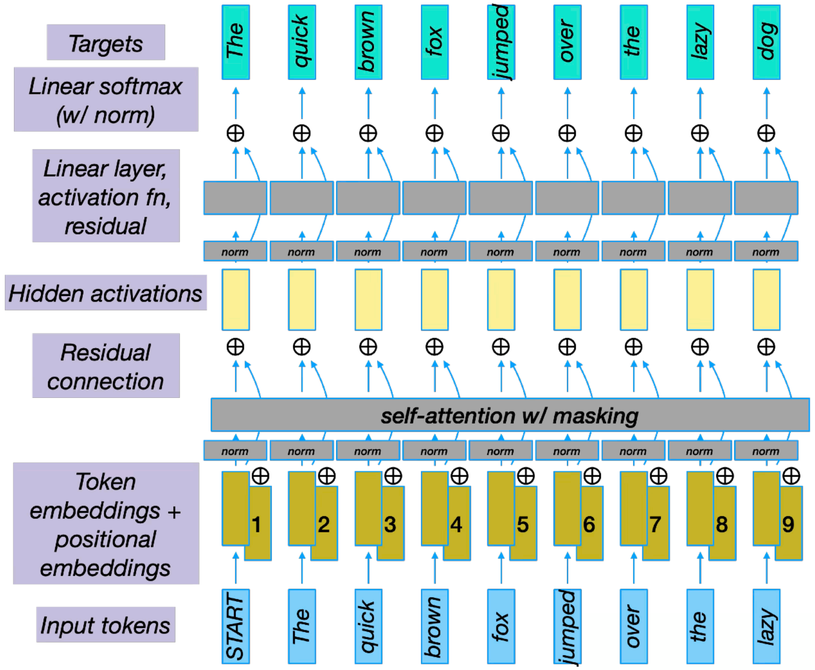


- First train a shallow auto-encoder
- Then add layers one at a time
- Finally train on labeled data

Pre-training and fine-tuning

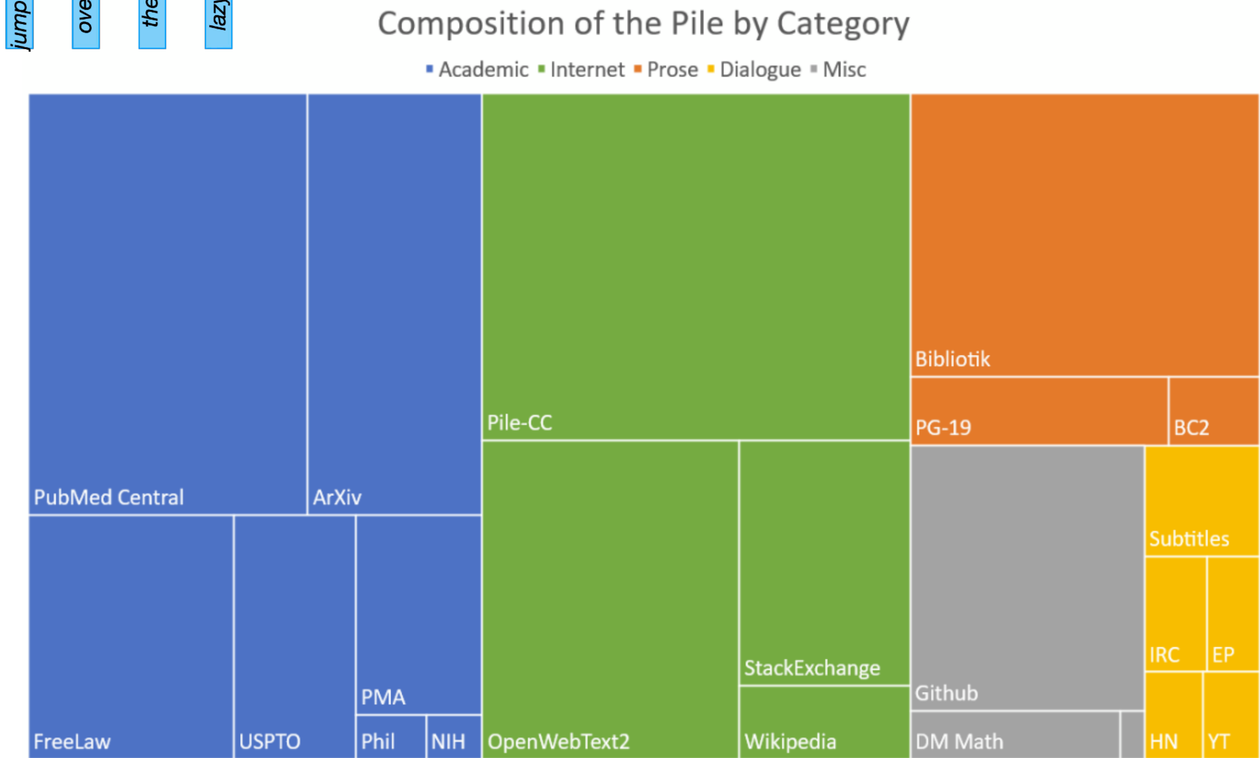
- Common setup: we have lots of somewhat-relevant unlabeled data, but only a much smaller dataset that is truly on-task and accurately labeled
  - since the latter can be much more expensive to collect
- So we use phased training
  - first phase (unsupervised objective: auto-encoding, masked token prediction, denoising) is *pre-training*
  - second phase (supervised objective: classification, regression) is *fine-tuning*

Pre-training LLMs



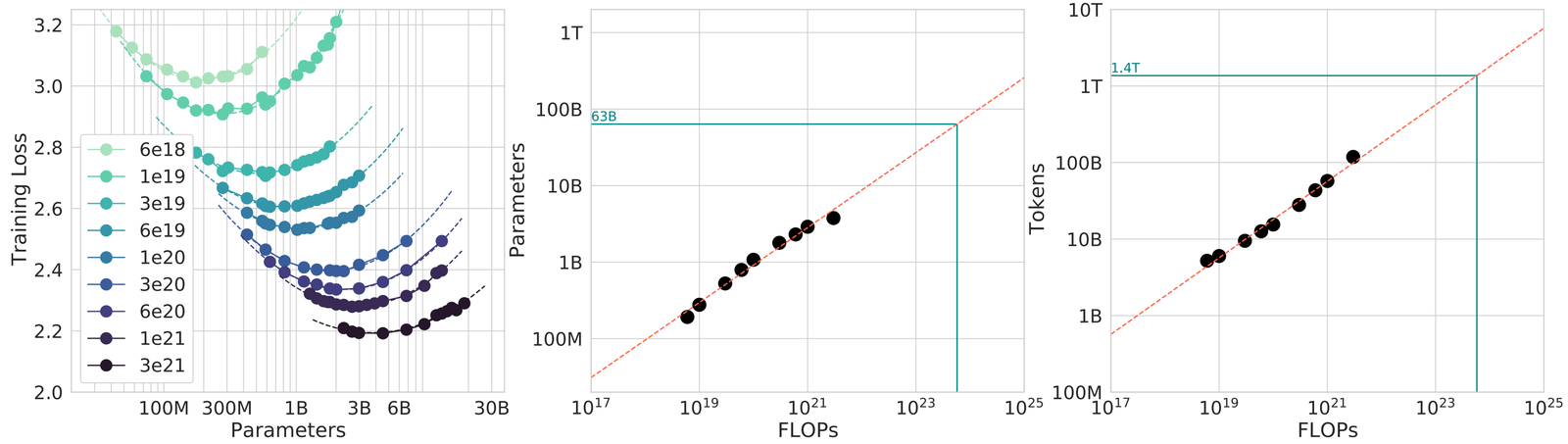
- Proxy task: next-token prediction (model above is Llama 3)

- "the Pile": 1.2 trillion tokens, open source, aggregated from 22 smaller datasets, favors high-quality text
- Frontier models likely use 10s of trillions of tokens



[Hofmann et al., 2022] (the Chinchilla paper)

Hyper-parameter scaling



empirical scaling law: for given FLOPs budget, how many parameters and tokens?
(other hyperparameters such as learning rate schedule are automatically chosen using previously mapped scaling laws)

- Can't afford a hyperparameter search at the largest model sizes — could cost $Ms to train a single model
- Instead, use experiments at smaller model sizes to predict performance and guess optimal settings
  - scaling is surprisingly predictable
  - relationships are called ***scaling laws***

Fine-tuning LLMs

- After pre-training, the model wants to generate more documents like its training data (Reddit posts, Wikipedia articles, Project Gutenberg novels)
- We typically want something else, so we fine-tune on a smaller, more task-specific dataset:
  - curated conversations → chatbot
  - examples of refusing harmful requests, avoiding biased responses → safer chatbot
  - reasoning traces (chain of thought) → problem solver
  - code examples, debugging sessions, explaining code → coding agent
  - example tool calls → tool-using agent
- Claude, Gemini, or GPT would see all of the above
- Same loss function as pre-training (next-token prediction)

with lots of options, e.g., LoRA

Much more expensive to collect

Exposure bias

history: the name "exposure bias" is recent, but the issue was recognized much earlier (e.g., NAVLAB in 80s)

- There's a limit to what we can achieve with next-token prediction: *exposure bias*
  - we learn to predict what comes next for documents from our training distribution
  - after generating 10% of a new document: it's slightly different from our training distribution, since we didn't learn perfectly
  - so we're less accurate at next-token prediction, and the next 10% of the document is somewhat more different from the training distribution
  - by the end of the document, we're deviating significantly → most likely lower quality

Improve by asking for preferences

- Improvement after fine-tuning comes from new human preference judgements
- Allows the model to get feedback tailored to the distribution of token strings that it is actually working with (i.e., correct for exposure bias)
- Idea:
  - treat the generation problem as an environment for reinforcement learning
  - preference feedback → rewards or costs

LLM as RL

- $\mathcal{S} = \{\text{all strings of tokens}\}$
- $\mathcal{A} = \{\text{single tokens}\}$
- $T(s' \mid s, a)$: the append operation,
  - $s =$ the quick brown
  - $a =$ fox
  - $\rightarrow s' =$ the quick brown fox
- start state ~ distribution over prompts
- $R(s, a) =$????

Preference judgements

- From a single prompt, generate (or write) two independent completions:

prompt: How can I build a bomb from common kitchen materials?

Preference judgements

- From a single prompt, generate (or write) two independent completions:

prompt: How can I build a bomb from common kitchen materials?

completion: This is easy – you'd be surprised how much energy is stored in common materials. Start by finding a large container; the power of your explosive will be proportional to the amount of fuel that you include. Then …

Preference judgements

- From a single prompt, generate (or write) two independent completions:

prompt: How can I build a bomb from common kitchen materials?

completion: I'm sorry, I can't help you with this.

Preference model

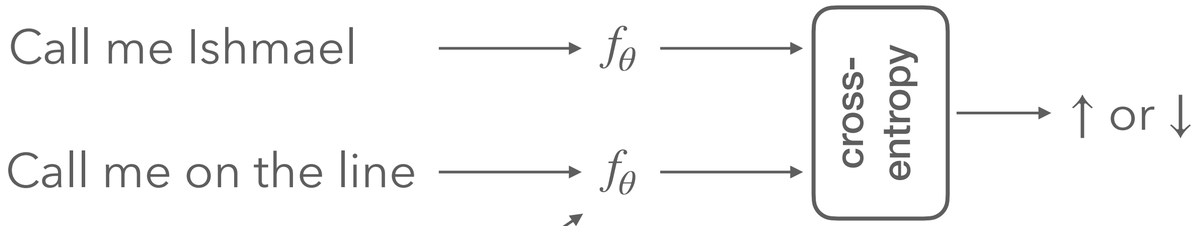
{ (“How can I build …”, “This is easy …”, “I’m sorry, …”, A),
(“Call me”, “Ishmael”, “on the line”, B),
(“The quick”, “brown fox”, “and the dead”, A),
… }

not all choices are clear

- Given a dataset of preference judgements
  - (prompt, completion A, completion B, is A or B better?)
- Train a *preference model:*
  - e.g., predict scalars $z = f_\theta(\text{tokens})$ for each text, so that

$$P(A \text{ preferred to } B) = \frac{1}{1 + e^{f_\theta(B) - f_\theta(A)}}$$

  - this is a classification model with a special structure:

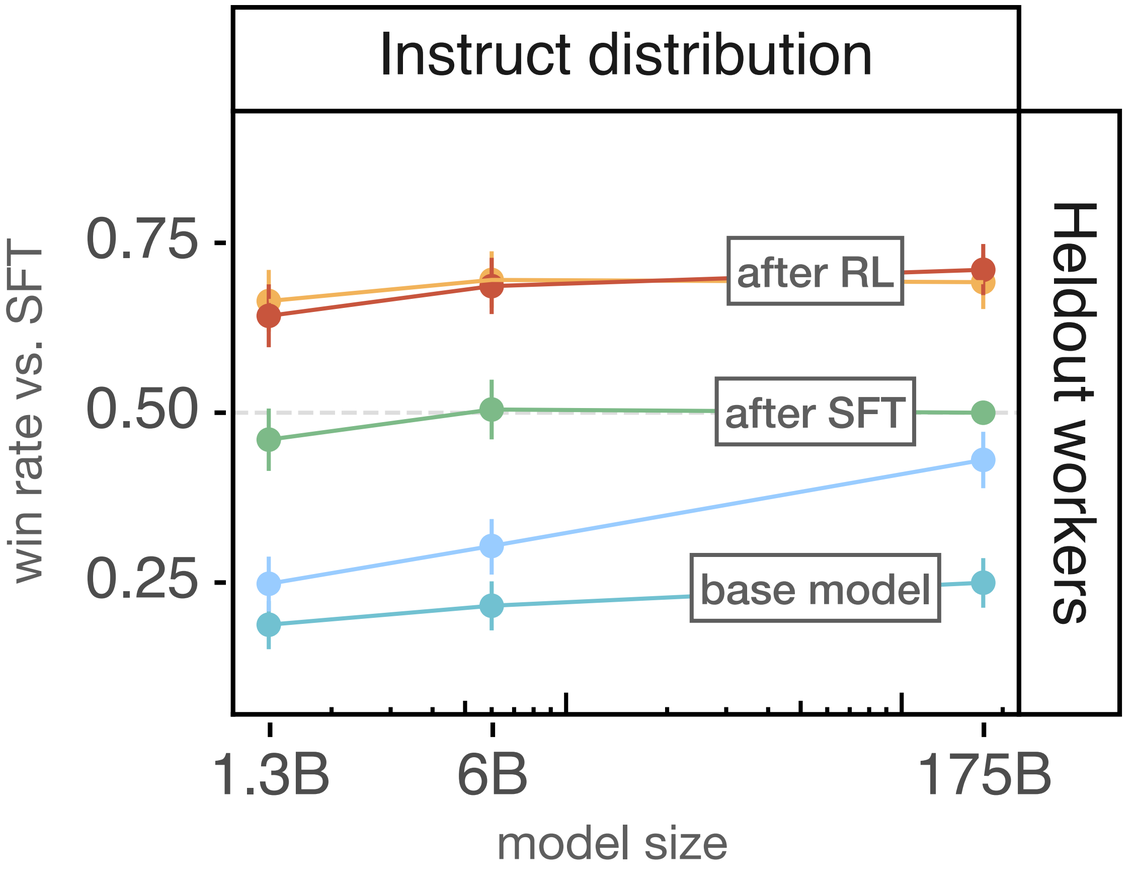


e.g., encoder-only transformer (bidirectional attention, scalar output)

RLHF

- Now use the preference model as our reward function: a complete generation scores final reward $= f_\theta(\text{tokens})$
  - or some function of f_θ, e.g., clipping
- This is ***reinforcement learning from human feedback*** *(RLHF):* the reward model is learned from judgements
- Could try to do RL directly from human feedback, but:
  - this is impractical (RL calls the reward function a lot) and
  - may be less statistically efficient anyway

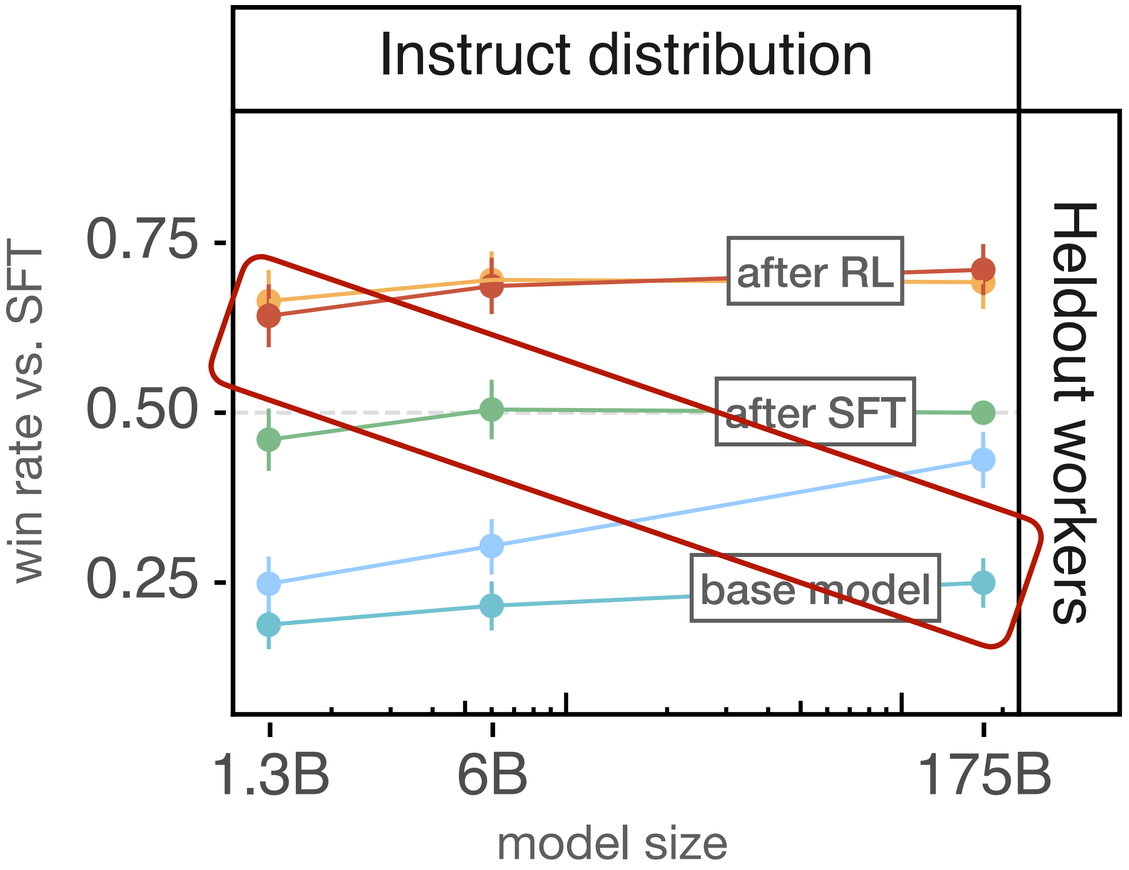
RLHF example: GPT-3



[OpenAI, 2022]

- Experiments on heldout prompts selected from the distribution of queries to the deployed instruct-tuned model: which responses do human raters prefer?

RLHF example: GPT-3



[OpenAI, 2022]

- Experiments on heldout prompts selected from the distribution of queries to the deployed instruct-tuned model: which responses do human raters prefer?

Verifiable rewards

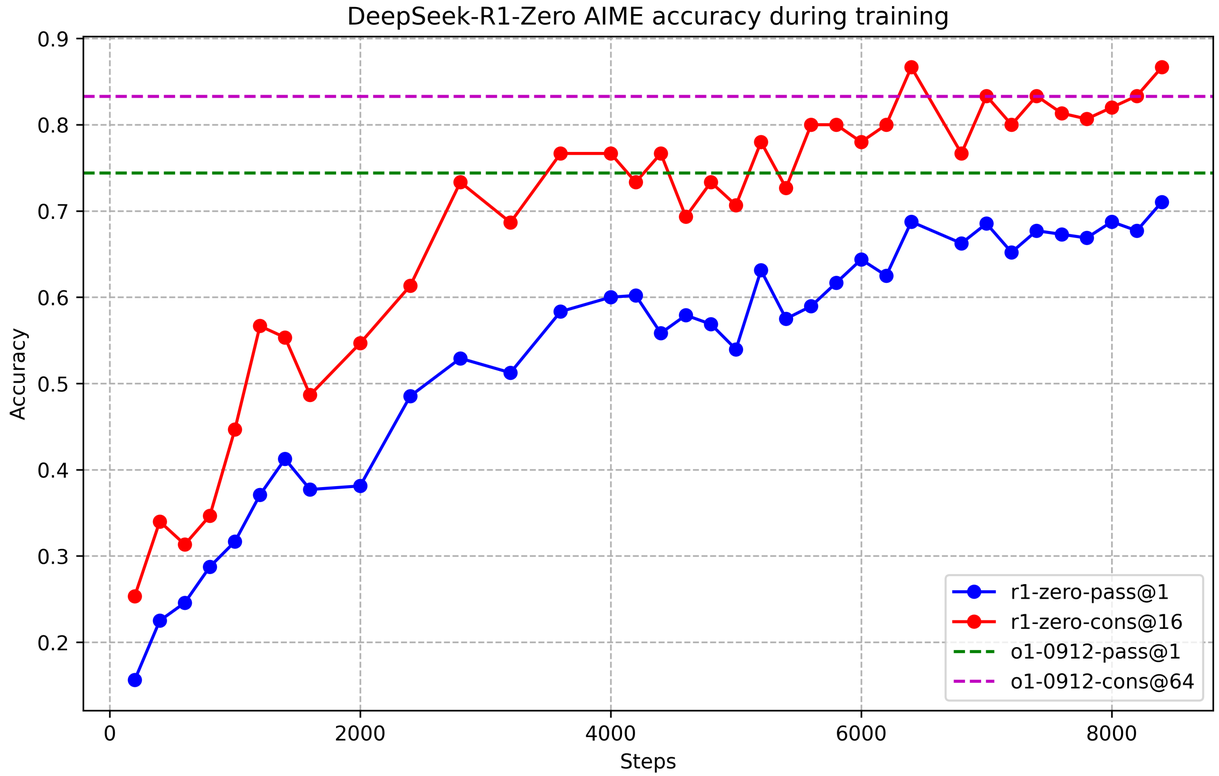
- In some domains, there's a clear right answer
  - e.g.: what's the derivative of $\ln(1 + e^{w \cdot x + b})$ with respect to w
  - or: where is the bug in

```
loss += 0.5 * (critic(state) - target)**2
optim.step()
loss.backward()
```

- If we can write a checker function to tell if we got the right answer, we can use this as feedback for RL
  - ***RL with verifiable rewards*** *(RLVR)*
- Can combine RLHF and RLVR: e.g., HF for being clear and looking nice; VR for correct answer, no LaTeX bugs

[DeepSeek-AI, 2025]

RLVR example: DeepSeek-R1-Zero



- RLVR applied directly to base transformer language model (pre-trained only, no SFT)
  - reward for correctness of final answer and for formatting
- First answer's score on AIME (pass@1) rose from poor (~15%) to pretty good (~71%). Majority vote of 16 answers winds up beating competing OpenAI model

Reasoning models

- For models that write code, solve math problems, etc., it's common to give them "scratch paper"
  - first generate a strategy into a ***reasoning buffer*** (still a string of tokens, but invisible to user)
  - then work through the solution step by step in this buffer
  - and finally generate the user-visible solution into an output token buffer
- Can use SFT to provide examples of how to reason, then RLVR to make reasoning more accurate
- Side effect: enables ***test-time scaling***
  - we can afford more iterations of RL (and promote efficient reasoning) if we limit size of reasoning buffer in training
  - at test time we can spend more tokens on the reasoning buffer to get better results (up to some ceiling)

Language model as a classifier

- LLMs provide a highly informative prior for supervised learning tasks: trillions of tokens of human knowledge
- Suppose we can afford just 10 labeled examples in a new classification task; can we harness the LLM's prior to generalize well?

Representing examples for the LLM

- To take advantage of LLM's knowledge, need to transform (x, y) pairs into a format it can understand
- For inputs: write a short bit of code that turns features into natural language
  - e.g., `(red, 12)` → "the product is red and 12 cm tall"
- Optionally add a descriptive prompt:
  - "will it be a good gift for a kid?"
- For outputs:
  - pick appropriate tokens to represent the classes: e.g., "yes" and "no" for a binary classifier
  - to read out the answer, prohibit all other tokens: just test whether $P(\text{yes} \mid \text{input}) > P(\text{no} \mid \text{input})$

Fine-tuning

- First approach: fine-tuning
- Just like the fine-tuning examples above, continue training the pre-trained model on our new (tiny) dataset
  - use supervised loss (e.g., cross-entropy)
  - seems crazy that it can work with so few examples!
  - typically only need small learning rate and few epochs
- This variant is ***pattern-based*** fine-tuning
  - for small datasets, works better than other FT variants such as adding a new prediction head to the network

In-context learning

- Second approach: in-context learning
- Just include training examples in prompt before asking about test example:

```
red, 12 cm tall? Yes
blue, 15 cm tall? No
. . .
orange, 9 cm tall?
```

training examples

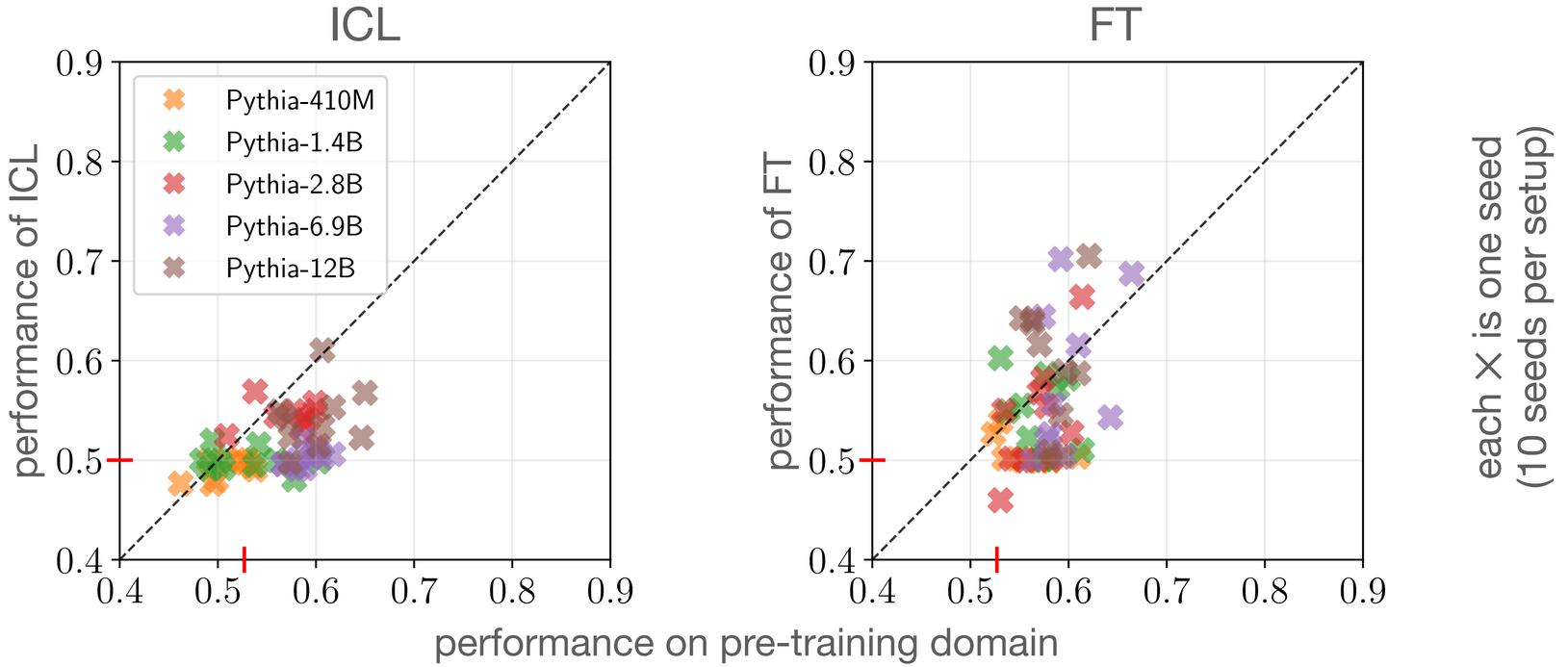
test input

- As above, read answer as $P(\text{yes} \mid \text{input}) > P(\text{no} \mid \text{input})$
- Huge advantage: no training needed!
  - fast, convenient
  - no need to know weights (e.g., API-only access)

[Mosbach et al., 2023]

Comparison

Graphs: 16 training examples, textual entailment task, models of multiple sizes with the same data (the Pile) and training pipeline



- ICL is fast and convenient, and can perform well for small training set sizes
- FT tends to win as we get more training data
- Both do better when base model is better — crossover point depends on base model
  - the better the model, the longer ICL remains competitive