1) Proof Practice

Question 1.1 (Example of Proof for Set Equality)
(From http://math.loyola.edu/ loberbro/ma421/BasicProofs.pdf) Let A and B be sets. If we want to prove that the sets are equal, i.e. $A = B$, then we must show two statements are true: $x \in A \implies x \in B$ and $x \in B \implies x \in A$. With this in mind, prove that the sets $A := \{n : 4k + 1 \text{ for some } k \in \mathbb{Z}\}$ and $B := \{n : 4k - 3 \text{ for some } k \in \mathbb{Z}\}$ are equal.

Solution. First we prove $x \in A \implies x \in B$. Let $x \in A$, and we know that $x = 4k + 1$ for some $k \in \mathbb{Z}$ (by the definition of A). If we can show that there exists $k' \in \mathbb{Z}$ such that $x = 4k' - 3$, then we will have shown $x \in B$ (by definition of B). We will do this by solving for k' and checking to see whether it is an integer. Using simple algebra to solve for k'...

$$\begin{aligned} x = 4k + 1 &= 4k' - 3 \\ k' &= \frac{1}{4}(4k + 4) \\ k' &= k + 1 \end{aligned}$$

Since $k, 1 \in \mathbb{Z}$ we know that $k + 1 \in \mathbb{Z}$ (by closure of integers under addition).

To complete the proof, we have to also show that $x \in B \implies x \in A$, although this argument looks nearly identical. This time, we assume $x \in B$, which means $\exists k \in \mathbb{Z}$ such that $x = 4k - 3$. To show $x \in A$ we want to show there is $k' \in \mathbb{Z}$ such that $x = 4k' + 1$. Solving for k'...

$$\begin{aligned} x = 4k - 3 &= 4k' + 1 \\ k' &= \frac{1}{4}(4k - 4) \\ k' &= k - 1 \end{aligned}$$

Again, we see that $k' \in \mathbb{Z}$, which means $x \in B$.

Question 1.2 (From *Book of Proof* by Richard Hammack) Consider integers $a, b, c \in \mathbb{Z}$. Show that if a is not a factor of bc, then a is not a factor of b. Hint: Prove by contrapositive. Recall a is a factor of b if there exists integer $x \in \mathbb{Z}$ such that $ax = b$.

Solution. By the hint, we prove contrapositive statement which is: if a is a factor of b, then a is a factor of bc. We start by assuming that a is a factor of b, implying that we can write $b = ax$. Then,

$$\begin{aligned} bc &= (ax)c && \text{By previous statement} \\ &= a(xc) && \text{By associative property of multiplication} \end{aligned}$$

Since $xc \in \mathbb{Z}$ (by closure of integers over multiplication), we see a is a factor of bc (based on definition of factor). Since we have proven the contrapositive statement is true, we have also shown the original statement is true.

Question 1.3 (From *Book of Proof* by Richard Hammack) For an integer $n \in \mathbb{Z}$, show that if n^2 is even then n is even. Hint: Prove by contrapositive.

Solution. Again, we prove the contrapositive statement: if n is odd, then n^2 is also odd. Assume that n is odd, and by definition of odd numbers, $\exists k \in \mathbb{Z}$ such that $n = 2k + 1$. Then

$$\begin{aligned} n^2 &= 4k^2 + 4k + 1 && \text{Using basic multiplication/addition properties.} \\ &= 2\left(2k^2 + 2k\right) + 1 && \text{Distributive property of multiplication.} \end{aligned}$$

Since $2k^2 + 2k \in \mathbb{Z}$, we see that n^2 is also odd. Since we have proven the contrapositive statement is true, we have also shown the original statement is true.

Question 1.4 (Challenge) (From *Book of Proof* by Richard Hammack) Prove that $\sqrt{2}$ is irrational. Hint: Prove this by contradiction by first assuming $\sqrt{2}$ is rational. That is, assume that there exists, $m, n \in \mathbb{Z}$ such that $\sqrt{2} = \frac{m}{n}$ and $\frac{m}{n}$ is in its simplest form, i.e. m and n share no factors. Then show that this cannot be the case. The fact we proved from question 1.4 will come in handy here.

Solution. We assume $\sqrt{2} = \frac{m}{n}$ as specified by the hint. Rearranging and squaring this equation we get $2n^2 = m^2$. Since $n^2 \in \mathbb{Z}$, we know that m^2 is an even number. Also, by the previous question we know that m is also an even number. In particular, we can write $m = 2k$, where $k \in \mathbb{Z}$. Then,

$$\begin{aligned} 2n^2 &= (2k)^2 \\ 2n^2 &= 4k^2 \\ n^2 &= 2k^2 \end{aligned}$$

So we see that n^2 is also an even number (since $k^2 \in \mathbb{Z}$). Again, using the previous question, n is also an even number. However, this is a contradiction because we assumed m and n have no common factors, but by definition of even numbers, they are both divisible by 2. Therefore, there cannot exist a valid m and n, and the statement is true by contradiction.

2) Propositional Logic (Taken from *How To Prove It*, by Daniel Velleman)

Question 2.1 Make a truth table for the following statements: $P \vee (Q \vee \neg P)$, $P \wedge \neg(Q \vee \neg Q)$, $P \vee \neg (Q \vee \neg Q)$

Solution.

P	Q	$P \vee (Q \vee \neg P)$	$P \wedge \neg(Q \vee \neg Q)$	$P \vee \neg (Q \vee \neg Q)$
F	F	T	F	F
T	F	T	F	T
F	T	T	F	F
T	T	T	F	T

Question 2.2 Show that $P \vee (Q \wedge \neg P)$ is equivalent to $P \vee Q$.

Solution.

$$P \vee (Q \wedge \neg P)$$
$$(P \vee Q) \wedge (P \vee \neg P) \qquad \text{(distributive law)}$$
$$P \vee Q \qquad \text{(tautology law, i.e. } P \vee \neg P \text{ is always true)}$$

Question 2.3 Show that $\neg (P \vee (Q \wedge \neg R)) \wedge Q$ is equivalent to $\neg P \wedge Q \wedge R$

Solution.

$$\neg (P \vee (Q \wedge \neg R)) \wedge Q$$
$$(\neg P \wedge \neg (Q \wedge \neg R)) \wedge Q \qquad \text{(DeMorgan's Law)}$$
$$(\neg P \wedge (\neg Q \vee \neg\neg R)) \wedge Q \qquad \text{(DeMorgan's Law)}$$
$$(\neg P \wedge (\neg Q \vee R)) \wedge Q \qquad \text{(double negation law)}$$
$$\neg P \wedge ((\neg Q \vee R) \wedge Q) \qquad \text{(associative law)}$$
$$\neg P \wedge (Q \wedge (\neg Q \vee R)) \qquad \text{(commutative law)}$$
$$\neg P \wedge ((Q \wedge \neg Q) \vee (Q \wedge R)) \qquad \text{(distributive law)}$$
$$\neg P \wedge (Q \wedge R) \qquad \text{(contradiction law, i.e. } Q \wedge \neg Q \text{ is always false)}$$

3) Perceptron Example

Recall the Perceptron algorithm we saw in class:

- Initialize $\mathbf{w} \leftarrow 0$
- Loop
  - Given point $\mathbf{x}$, predict $\hat{y} = \text{sign}(\mathbf{w}^T\mathbf{x})$
  - Given actual label y, if $y \neq \hat{y} \ldots$
    - If $y = +1$, $\mathbf{w} \leftarrow \mathbf{w} + \mathbf{x}$
    - If $y = -1$, $\mathbf{w} \leftarrow \mathbf{w} - \mathbf{x}$

You can assume that $\text{sign}(0) = 1$. For the following, let $\mathbf{x}, \mathbf{w} \in \mathbb{R}^2$ and we are given a dataset made up of the following four points:

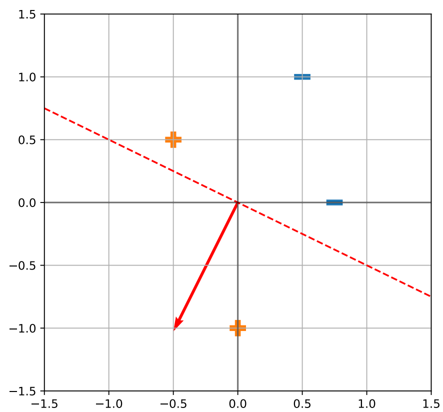
$$\begin{aligned} \mathbf{x}_1 &= [1/2, 1]^T & y_1 &= -1 \\ \mathbf{x}_2 &= [3/4, 0]^T & y_2 &= -1 \\ \mathbf{x}_3 &= [0, -1]^T & y_3 &= 1 \\ \mathbf{x}_4 &= [-1/2, 1/2]^T & y_4 &= 1 \end{aligned}$$

Do the first loop of the perceptron algorithm by hand. That is, for each $(\mathbf{x}_1, y_1), \ldots, (\mathbf{x}_4, y_4)$ do the following:

- Write down the prediction $\hat{y}$.
- Write down the updated vector $\mathbf{w}$
- Draw the updated vector $\mathbf{w}$ on a plot below.
- Draw the decision boundary line $0 = \mathbf{w}^T\mathbf{x}$.

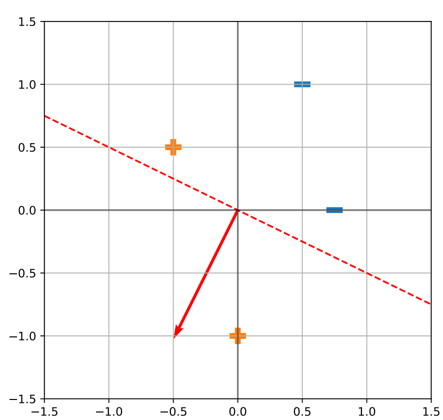
Data Point $(\mathbf{x}_1, y_1)$

- $\hat{y} = \text{sign}\left(0 \times \frac{1}{2} + 0 \times 1\right) = \text{sign}(0) = 1$
- $\mathbf{w} = [0, 0]^T - [1/2, 1]^T = [-1/2, -1]^T$



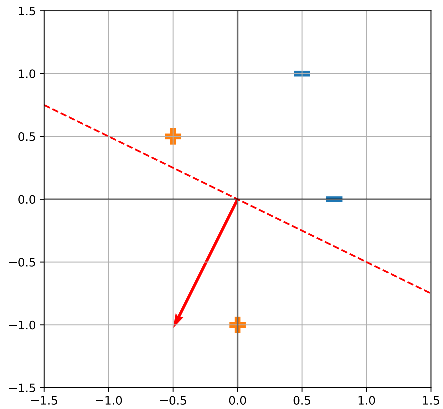
Data Point $(\mathbf{x}_2, y_2)$

- $\hat{y} = \text{sign}\left(\frac{-1}{2} \times \frac{1}{2} - 1 \times 0\right) = \text{sign}(\frac{-1}{4}) = -1$
- $\mathbf{w} = [-1/2, -1]^T$(no update)



Data Point $(\mathbf{x}_3, y_3)$

- $\hat{y} = \text{sign}\left(\frac{-1}{2} \times 0 - 1 \times -1\right) = \text{sign}(1) = 1$
- $\mathbf{w} = [-1/2, -1]^T$(no update)



Data Point $(\mathbf{x}_4, y_4)$

- $\hat{y} = \text{sign}\left(\frac{-1}{2} \times \frac{-1}{2} - 1 \times \frac{1}{2}\right) = \text{sign}(\frac{-1}{4}) = -1$
- $\mathbf{w} = [-1/2, -1]^T + [-1/2, 1/2]^T = [-1, -1/2]^T$