1) Proof Practice

Question 1.1 (Example of Proof for Set Equality)
(From http://math.loyola.edu/ loberbro/ma421/BasicProofs.pdf) Let A and B be sets. If we want to prove that the sets are equal, i.e. $A = B$, then we must show two statements are true: $x \in A \implies x \in B$ and $x \in B \implies x \in A$. With this in mind, prove that the sets $A := \{n : 4k + 1 \text{ for some } k \in \mathbb{Z}\}$ and $B := \{n : 4k - 3 \text{ for some } k \in \mathbb{Z}\}$ are equal.

Question 1.2 (From *Book of Proof* by Richard Hammack) Consider integers $a, b, c \in \mathbb{Z}$. Show that if a is not a factor of bc, then a is not a factor of b. Hint: Prove by contrapositive. Recall a is a factor of b if there exists integer $x \in \mathbb{Z}$ such that $ax = b$.

Question 1.3 (From *Book of Proof* by Richard Hammack) For an integer $n \in \mathbb{Z}$, show that if n^2 is even then n is even. Hint: Prove by contrapositive.

Question 1.4 (Challenge) (From *Book of Proof* by Richard Hammack) Prove that $\sqrt{2}$ is irrational. Hint: Prove this by contradiction by first assuming $\sqrt{2}$ is rational. That is, assume that there exists, $m, n \in \mathbb{Z}$ such that $\sqrt{2} = \frac{m}{n}$ and $\frac{m}{n}$ is in its simplest form, i.e. m and n share no factors. Then show that this cannot be the case. The fact we proved from question 1.3 will come in handy here.

2) Propositional Logic (Taken from *How To Prove It*, by Daniel Velleman)

Question 2.1 Make a truth table for the following statements: $P \vee (Q \vee \neg P)$, $P \wedge \neg(Q \vee \neg Q)$, $P \vee \neg (Q \vee \neg Q)$

Question 2.2 Show that $P \vee (Q \wedge \neg P)$ is equivalent to $P \vee Q$.

Question 2.3 Show that $\neg (P \vee (Q \wedge \neg R)) \wedge Q$ is equivalent to $\neg P \wedge Q \wedge R$

3) Perceptron Example

Recall the Perceptron algorithm we saw in class:

- Initialize $\mathbf{w} \leftarrow 0$
- Loop
  - Given point $\mathbf{x}$, predict $\hat{y} = \text{sign}(\mathbf{w}^T\mathbf{x})$
  - Given actual label y, if $y \neq \hat{y} \ldots$
    - If $y = +1$, $\mathbf{w} \leftarrow \mathbf{w} + \mathbf{x}$
    - If $y = -1$, $\mathbf{w} \leftarrow \mathbf{w} - \mathbf{x}$

You can assume that $\text{sign}(0) = 1$. For the following, let $\mathbf{x}, \mathbf{w} \in \mathbb{R}^2$ and we are given a dataset made up of the following four points:

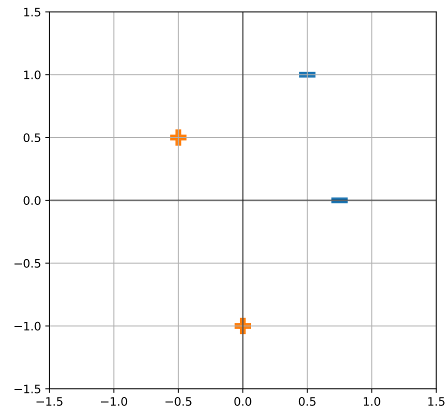
$$\begin{aligned} \mathbf{x}_1 &= [1/2, 1]^T & y_1 &= -1 \\ \mathbf{x}_2 &= [3/4, 0]^T & y_2 &= -1 \\ \mathbf{x}_3 &= [0, -1]^T & y_3 &= 1 \\ \mathbf{x}_4 &= [-1/2, 1/2]^T & y_4 &= 1 \end{aligned}$$

Do the first loop of the perceptron algorithm by hand. That is, for each $(\mathbf{x}_1, y_1), \ldots, (\mathbf{x}_4, y_4)$ do the following:

- Write down the prediction $\hat{y}$.
- Write down the updated vector $\mathbf{w}$
- Draw the updated vector $\mathbf{w}$ on a plot below.
- Draw the decision boundary line $0 = \mathbf{w}^T\mathbf{x}$.

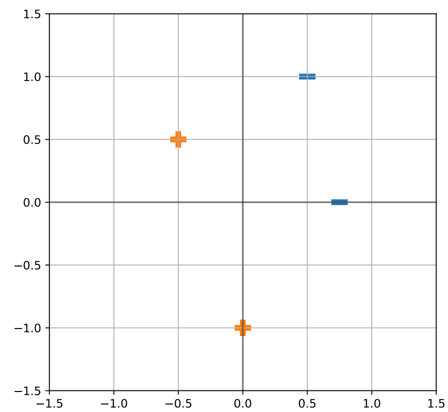
Data Point $(\mathbf{x}_1, y_1)$

- $\hat{y} =$
- $\mathbf{w} =$



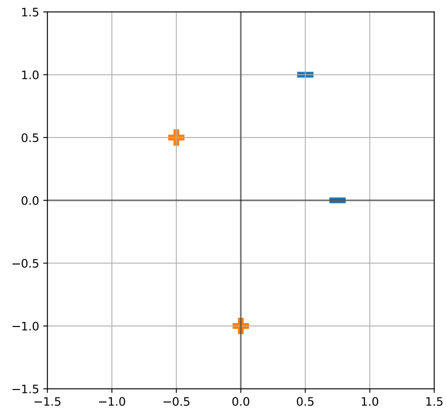
Data Point $(\mathbf{x}_2, y_2)$

- $\hat{y} =$
- $\mathbf{w} =$



Data Point $(\mathbf{x}_3, y_3)$

- $\hat{y} =$
- $\mathbf{w} =$



Data Point $(\mathbf{x}_4, y_4)$

- $\hat{y} =$
- $\mathbf{w} =$