Mathematical Foundations for Machine Learning

Linear Algebra and Linear Regression

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Today

Linear algebra

- Primer and highlighting key aspects

Linear regression

- Calculus and optimization

Linear Algebra

Linear Algebra

Convenient and concise way to work with math involving linear operations

$$3\theta_1 + 0\theta_2 + 5\theta_3 = 36$$
$$4\theta_1 + 3\theta_2 + 2\theta_3 = 46$$
$$2\theta_1 + 2\theta_2 + 1\theta_3 = 25$$

$$V\boldsymbol{\theta} = \boldsymbol{u} \qquad V = \begin{bmatrix} 3 & 0 & 5 \\ 4 & 3 & 2 \\ 2 & 2 & 1 \end{bmatrix} \qquad \boldsymbol{u} = \begin{bmatrix} 36 \\ 46 \\ 25 \end{bmatrix} \qquad \boldsymbol{\theta} = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix}$$

Possible way to solve: $\boldsymbol{\theta} = V^{-1}\boldsymbol{u}$

(Actually a pretty unstable way to solve in general. More on this later.)

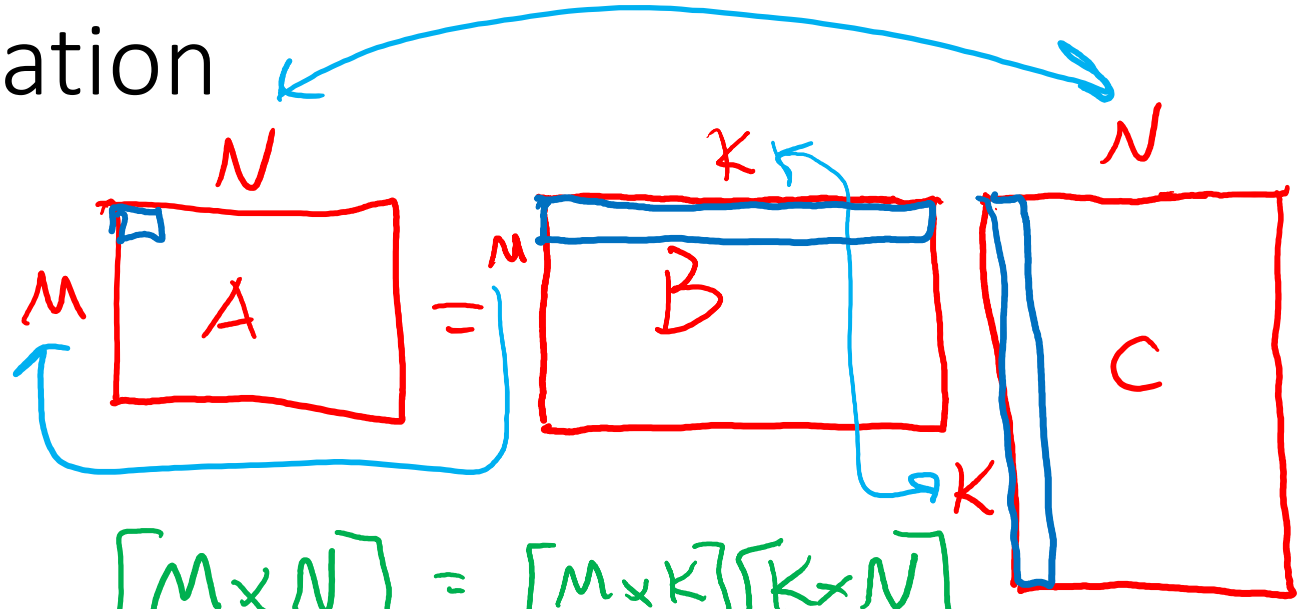
Matrix Multiplication

$A \in \mathbb{R}^{M\times N}$

$B \in \mathbb{R}^{M\times K}$

$C \in \mathbb{R}^{K\times N}$

$A = BC$



$[M \times N] = [M \times K][K \times N]$

The entry of A in row m and column n is:

$$A_{m,n} = \sum_{k=1}^{K} B_{m,k} C_{k,n}$$

i.e. multiply the elements of row m of by the elements of column n of C, and then sum those products

Width of B must match height of C

Poll 1

What is the resulting dimension of Z, where $Z = \boldsymbol{u}\boldsymbol{v}^T$ and $\boldsymbol{u} \in \mathbb{R}^{7\times1}$ and $\boldsymbol{v} \in \mathbb{R}^{7\times1}$?

Select ALL that apply

$A.\ \ Z \in \mathbb{R}$

$B.\ \ Z \in \mathbb{R}^{49}$

$C.\ \ Z \in \mathbb{R}^{7\times1}$

$D.\ \ Z \in \mathbb{R}^{1\times7}$

$E.\ \ Z \in \mathbb{R}^{1\times1}$

$F.\ \ Z \in \mathbb{R}^{7\times7}$ 95%

G. I have no idea

Vectors

$\mathbf{u} \in \mathbb{R}^{M \times 1}$ Column vector of length M

$\mathbf{v} \in \mathbb{R}^{1 \times M}$ Row vector of length M

$\mathbf{z} \in \mathbb{R}^{M}$ Ambiguous. In this course, assume column vector unless stated otherwise.

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_M \end{bmatrix}$$

Vectors

$\mathbf{u} \in \mathbb{R}^M$

$\mathbf{v} \in \mathbb{R}^M$ Note these are both column vectors

Multiplication of vectors of the same length

Dot product (inner product):

- $y = \mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T \mathbf{v} = \sum_{i=1}^{M} u_i \, v_i \qquad y \in \mathbb{R}$
- A dot product is an inner product (inner product is a more general term)

Outer product (~~inner product~~):

- $Y = \mathbf{u} \otimes \mathbf{v} = \mathbf{u}\mathbf{v}^T \qquad Y \in \mathbb{R}^{M \times M} \qquad \mathrm{Y}_{i,j} = u_i v_j$

Vectors

$\mathbf{u} \in \mathbb{R}^M$

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \rightarrow \text{sum of squares}$$

What happens when we do the dot product of a vector with itself?

$$\vec{u}^T u = \sum_{i=1}^{M} u_i^2$$

Vectors

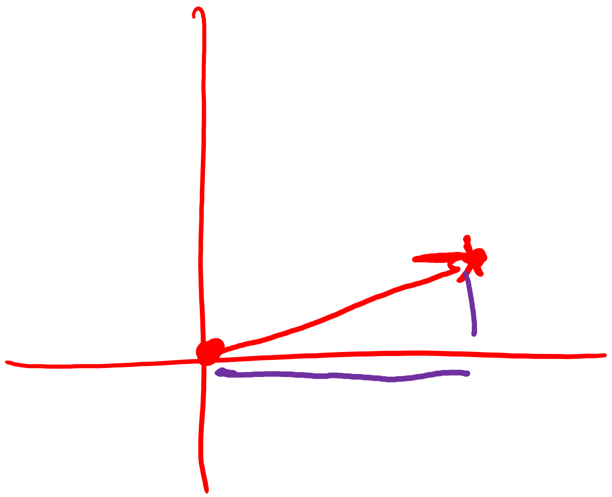
$\mathbf{u} \in \mathbb{R}^M$

Vector magnitude

$u^T u$

$u \cdot u$

$$|\mathbf{u}| = \sqrt{\textstyle\sum_i^M u_i^2} = \left(\textstyle\sum_i^M u_i^2\right)^{1/2} = (\ ?\ )^{1/2}$$



L2 norm (Euclidean norm)

(More norms later)

$$\|\mathbf{u}\|_2 = \sqrt{\textstyle\sum_i^M u_i^2} = \left(\textstyle\sum_i^M u_i^2\right)^{1/2} = \left(\boldsymbol{u}^T \boldsymbol{u}\right)^{1/2}$$

L2 norm squared

$$\|\mathbf{u}\|_2^2 = \textstyle\sum_i^M u_i^2 = \boldsymbol{u}^T \boldsymbol{u}$$

Vectors

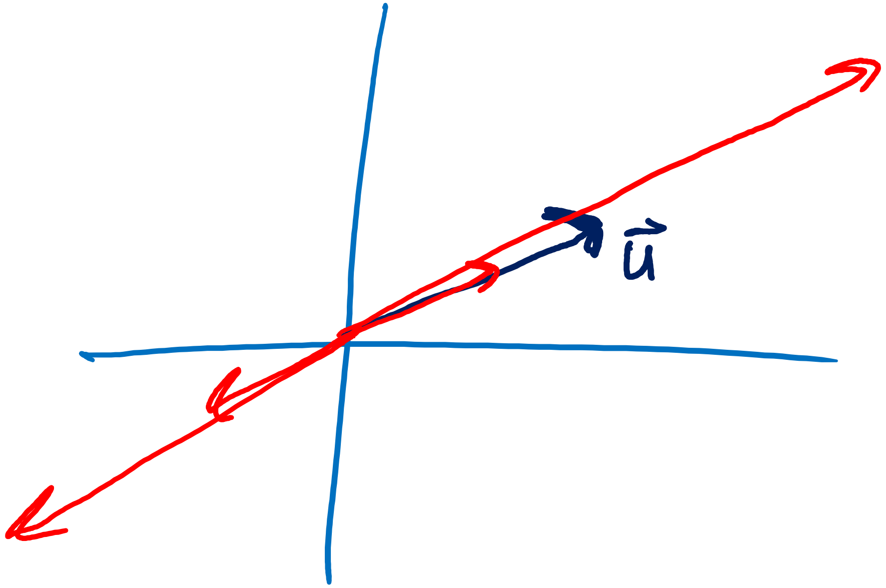
$\mathbf{u} \in \mathbb{R}^M$

$\mathbf{v} \in \mathbb{R}^M$

$\alpha \in \mathbb{R}$

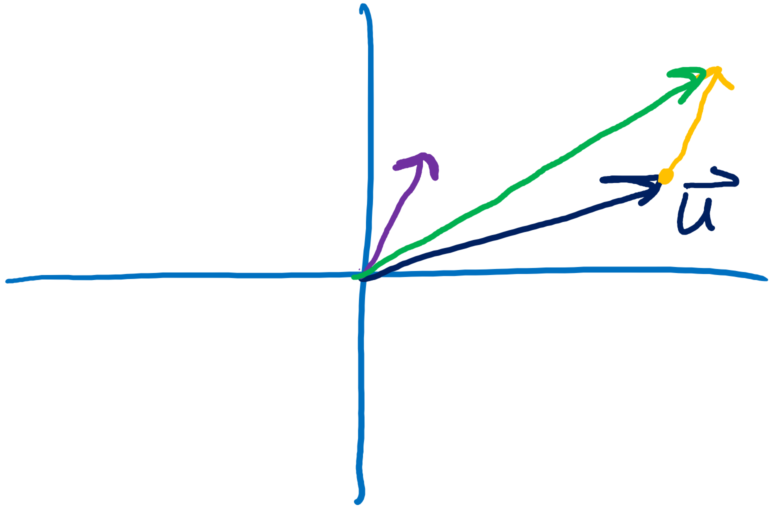
Scalar multiplication

$$\alpha \mathbf{u} = \begin{bmatrix} \alpha u_1 \\ \alpha u_2 \\ \vdots \\ \alpha u_M \end{bmatrix}$$



Vector addition

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_M + v_M \end{bmatrix}$$



Notation alert!

Vectors

$\mathbf{u}, \mathbf{v} \in \mathbb{R}^M$

Linear combination of vectors

$$\mathbf{z} = \alpha\mathbf{u} + \beta\mathbf{v} = \begin{bmatrix} \alpha u_1 + \beta v_1 \\ \alpha u_2 + \beta v_2 \\ \vdots \\ \alpha u_M + \beta v_M \end{bmatrix} \text{ for any } \alpha, \beta \in \mathbb{R}$$

Span

Notation alert!

Set of all linear combination of a set of vectors

Given a set of vectors $\mathcal{S} = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$, where $\boldsymbol{v}_i \in \mathbb{R}^M\ \forall\ i$

$$span(\mathcal{S}) = \{ w_1\boldsymbol{v}_1 + w_2\boldsymbol{v}_2 + w_3\boldsymbol{v}_3 \mid w_i \in \mathbb{R} \}$$

Span of a set of vectors is an example of a vector space. Vector space is a more general term.

Span, Dimensionality, and Rank

$$V = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & v_3 \\ \downarrow & \downarrow & \downarrow \end{bmatrix}$$

Rank$(V) \longrightarrow$ dimension of vector space spanned by $\{v_1, v_2, v_3\}$

$$\uparrow \quad \uparrow \quad \uparrow$$

Span, Dimensionality, and Rank

$A \in \mathbb{R}^{5\times 4}$

$$\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \end{bmatrix}$$

$A' \in \mathbb{R}^{4\times 5}$

$$\begin{bmatrix} a_1^T \\ a_2^T \\ a_3^T \\ a_4^T \end{bmatrix}$$

Linear Regression

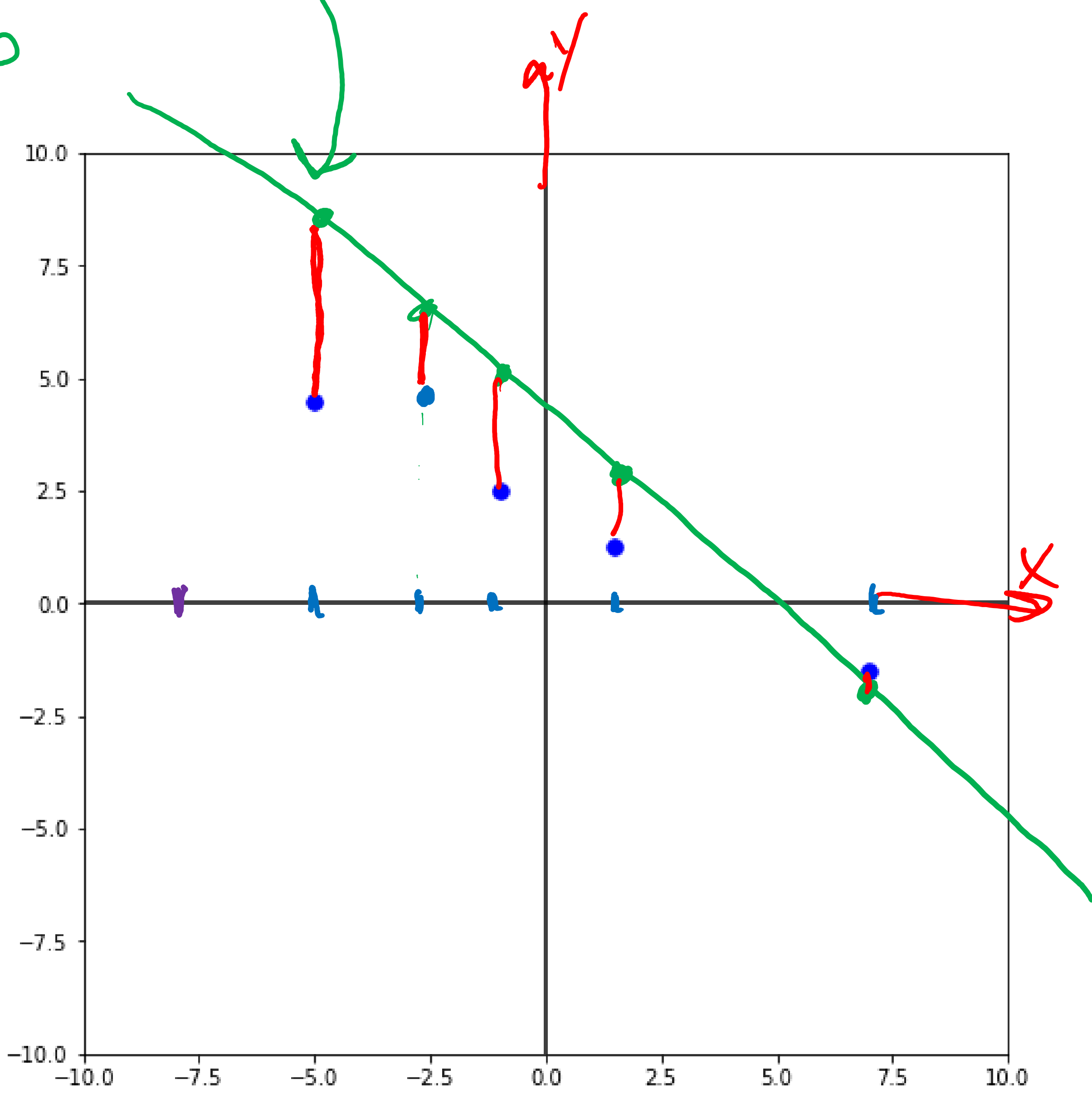
Fitting a linear model

$$\mathcal{D} = \{(x^{(i)}, y^{(i)})\}_{i=1}^{4}$$
$$= \{(-1,\ \ 2.5),$$
$$(7,\ -1.5),$$
$$(-5,\ 4.5),$$
$$(1.5,\ 1.25)\}$$

$$\hat{y} = mx + b$$

$$x_{new} = -8$$

$$\sum_{i=1}^{N} \Big(\underbrace{y^{(i)} - \hat{y}^{(i)}}_{\text{error}}\Big)^2$$



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