As you walk in

Welcome!

1) Sit at a table next to another student
2) Make name plate
   - Fold paper in half
   - Write preferred name

Mathematical Foundations for Machine Learning

Linear Systems

Instructor: Pat Virtue

Today

Linear Systems

- Systems of equations
- Fitting linear models to data

Detailed course topics doc

https://docs.google.com/spreadsheets/d/1I8g9pR-krQNKVuiPrxGUEkSwdyDjBIpTr9inq6Dl8-U

Exercise

Given the following system of equations, *how* would you solve for acceptable values for $\theta_1, \theta_2, \theta_3$? (Solve it too ☺)

Alien coins! Your friend E.T. is helping you to learn alien currency. There are three different types of coins that have values $\theta_1, \theta_2, \theta_3$. There are three different piles of coins. E.T. is kind enough to tell us the total value of each of the three piles.

$$
\begin{aligned}
3\theta_1 + 0\theta_2 + 5\theta_3 &= 36 \\
4\theta_1 + 3\theta_2 + 2\theta_3 &= 46 \\
2\theta_1 + 2\theta_2 + 1\theta_3 &= 25
\end{aligned}
$$

Exercise

Given the following system of equations, how would you solve for acceptable values for $\theta_1, \theta_2, \theta_3$?

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Notation alert!

Linear Algebra

Linear algebra allows us to represent and operate upon sets of linear equations.

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$$4\theta_1 + 3\theta_2 + 2\theta_3 = 46$$
$$2\theta_1 + 2\theta_2 + 1\theta_3 = 25$$

$$V\boldsymbol{\theta} = \boldsymbol{u} \qquad V = \begin{bmatrix} 3 & 0 & 5 \\ 4 & 3 & 2 \\ 2 & 2 & 1 \end{bmatrix} \quad \boldsymbol{u} = \begin{bmatrix} 36 \\ 46 \\ 25 \end{bmatrix} \quad \boldsymbol{\theta} = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix}$$

Possible way to solve: $\boldsymbol{\theta} = V^{-1}\boldsymbol{u}$

(Actually a pretty unstable way to solve in general. More on this later.)

Poll 1

What could happen if we added one more equation?

$$3\theta_1 + 0\theta_2 + 5\theta_3 = 36$$
$$4\theta_1 + 3\theta_2 + 2\theta_3 = 46$$
$$2\theta_1 + 2\theta_2 + 1\theta_3 = 25$$

Select ALL that apply

A. No solution

B. One solution

C. Two solutions

D. Infinite solutions

E. Error

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Select ALL that apply

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B. One solution

C. Two solutions

D. Infinite solutions

E. Error

Poll 2

What could happen if we remove one of these equations?

$$3\theta_1 + 0\theta_2 + 5\theta_3 = 36$$
$$4\theta_1 + 3\theta_2 + 2\theta_3 = 46$$
$$2\theta_1 + 2\theta_2 + 1\theta_3 = 25$$

Select ALL that apply

A. No solution

B. One solution

C. Two solutions

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E. Error

Poll 2

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$$3\theta_1 + 0\theta_2 + 5\theta_3 = 36$$
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Notation alert!

Exercise

Write a system of equations to help us fit a line to the following data:

$\mathcal{D} = \left\{\left(x^{(i)}, y^{(i)}\right)\right\}_{i=1}^{N}$, where N is the number of points in the dataset

Specifically:

$$\mathcal{D} = \{(-1,\ \ 2.5),\\ (7,\ -1.5),\\ (-5,\ 4.5),\\ (1.5,\ 1.25)\}$$

Notation alert!

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Write a system of equations to help us fit a line to the following data:

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Specifically:

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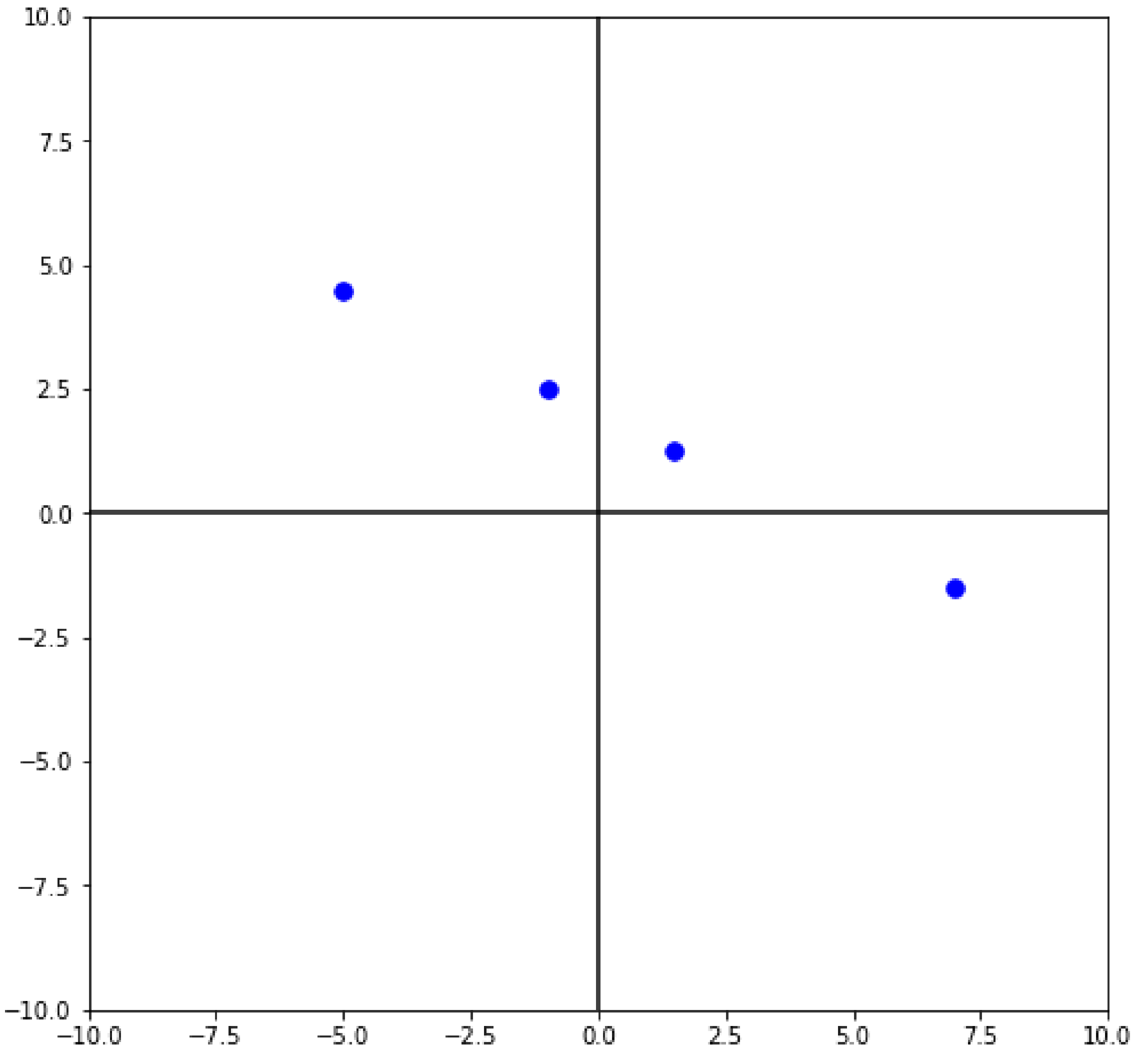
Poll 3

Can we fit a linear model to this data?

$$\mathcal{D} = \{(x^{(i)}, y^{(i)})\}_{i=1}^{4}$$
$$= \{(-1,\ \ 2.5),$$
$$(7,\ \ -1.5),$$
$$(-5,\ \ 4.5),$$
$$(1.5,\ \ 1.25)\}$$



Linear and Affine

Notation alert!
$\forall$
$\mathbb{R}^M$

Linear combination (of a set of terms)

Multiplying each term by a scalar and adding the results

e.g. Given a set of terms $\mathcal{S} = \{x_1, x_2, x_3\}$, where $x_i \in \mathbb{R} \ \forall \ i \in \{1 \dots 3\}$
$w_1 x_1 + w_2 x_2 + w_3 x_3$ is a linear combination of $\mathcal{S}$ if $w_i \in \mathbb{R} \ \forall \ i \in \{1 \dots 3\}$

e.g. Given a set of terms $\mathcal{S} = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$, where $\boldsymbol{v}_i \in \mathbb{R}^M \ \forall \ i \in \{1 \dots 3\}$
$w_1 \boldsymbol{v}_1 + w_2 \boldsymbol{v}_2 + w_3 \boldsymbol{v}_3$ is a linear combination of $\mathcal{S}$ if $w_i \in \mathbb{R} \ \forall \ i \in \{1 \dots 3\}$

Linear and Affine

Affine combination (of a set of terms)

Affine allows for an additional scalar term to be added to a linear combination. Often called an offset or bias term

$w_1 x_1 + w_2 x_2 + w_3 x_3 + b$, where $b \in \mathbb{R}$

$w_1 \boldsymbol{v}_1 + w_2 \boldsymbol{v}_2 + w_3 \boldsymbol{v}_3 + b$, where $b \in \mathbb{R}$

Linear vs Affine Models

What linear usually means depends on the domain:

Linear algebra:

Linear usually means strictly linear

Geometry, algebra:

Linear usually means affine

Machine learning:

Linear usually means affine but we often transform affine to strictly linear to make the linear algebra and notation easier

Poll 3

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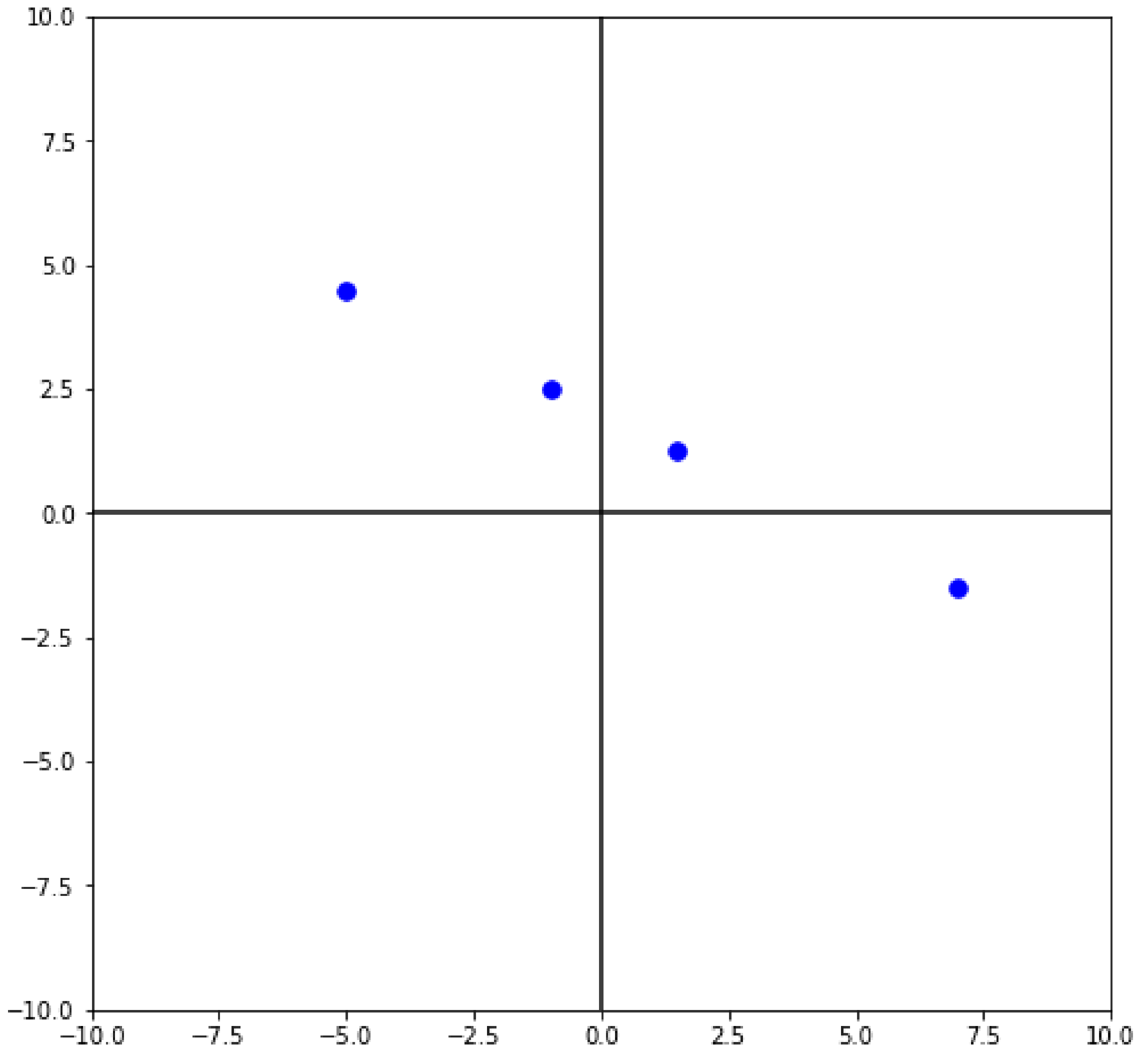

Linear (but not affine) model will include the origin

We don't need an offset (or bias) term to fit this data

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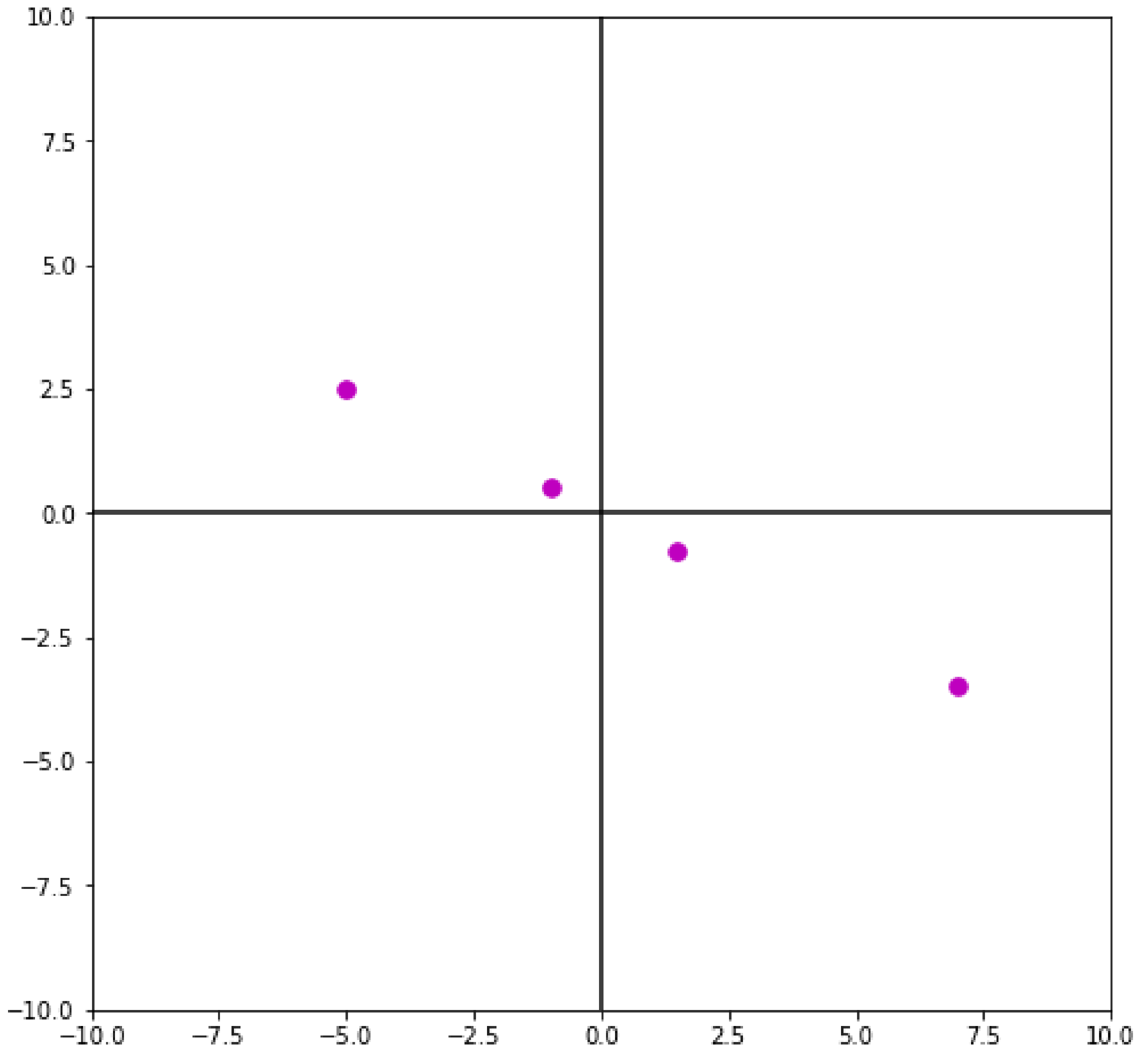
$$= \{(-1,\ 0.5),$$
$$(7,\ -3.5),$$
$$(-5,\ 2.5),$$
$$(1.5,\ -0.75)\}$$



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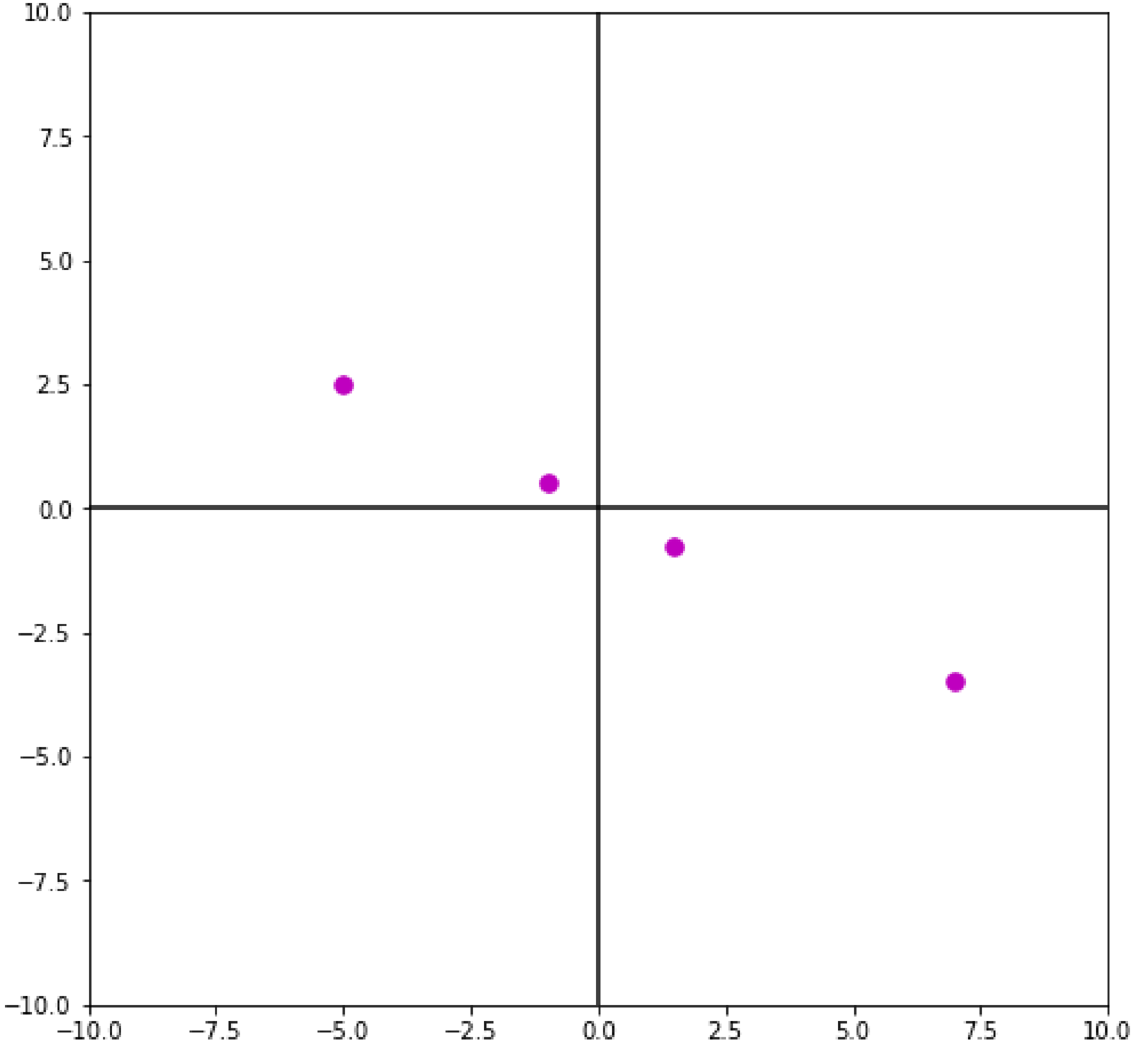

Linear (Affine) in Higher Dimensions

What are these linear shapes called for 1-D, 2-D, 3-D, M-D input?

$\boldsymbol{x} \in \mathbb{R}$ $\quad \boldsymbol{x} \in \mathbb{R}^2$ $\quad \boldsymbol{x} \in \mathbb{R}^3$ $\quad \boldsymbol{x} \in \mathbb{R}^M$

$$y = \boldsymbol{w}^T \boldsymbol{x} + b$$