1 Lagrange Multipliers for PCA

Consider the following optimization where A is a symmetric matrix:

$$\hat{\mathbf{v}} = \underset{\mathbf{v}}{\text{argmax.}} \ \mathbf{v}^T A \mathbf{v}$$
$$\text{s.t.} \ \|\mathbf{v}\|_2^2 = 1$$

1. Formulate the Lagrangian, $\mathcal{L}(\mathbf{v}, \lambda)$.

Lagrangian

Solution

$$\mathcal{L}(\mathbf{v}, \lambda) = \mathbf{v}^T A \mathbf{v} - \lambda(\|\mathbf{v}\|_2^2 - 1)$$

2. Write the gradient of the Lagrangian with respect to $\mathbf{v}$, $\nabla_{\mathbf{v}} \mathcal{L}(\mathbf{v}, \lambda)$.

$\nabla_{\mathbf{v}} \mathcal{L}$

Solution

$$\mathcal{L}(\boldsymbol{\theta}, \lambda) = \mathbf{v}^T A \mathbf{v} - \lambda(\mathbf{v}^T \mathbf{v} - 1)$$

$$\nabla_{\boldsymbol{\theta}} \mathcal{L}(\boldsymbol{\theta}, \lambda) = 2A\mathbf{v} - 2\lambda\mathbf{v}$$

3. After setting $\nabla_{\mathbf{v}} \mathcal{L}(\mathbf{v}, \lambda)$ equal to zero, what can you say about the relationship between λ, $\mathbf{v}$, and the eigenvalues and eigenvectors of A?

Eigenvectors

Solution

$$2A\mathbf{v} - 2\lambda\mathbf{v} = 0$$
$$A\mathbf{v} = \lambda\mathbf{v}$$

$\mathbf{v}$ is an eigenvector of A and λ is the corresponding eigenvalue of A!