

Warm-up

1. https://www.sporcle.com/games/MrChewypoo/minimalist_disney
2. <https://www.sporcle.com/games/Stanford0008/minimalist-cartoons-slideshow>
3. <https://www.sporcle.com/games/MrChewypoo/minimalist>

Announcements

HW3

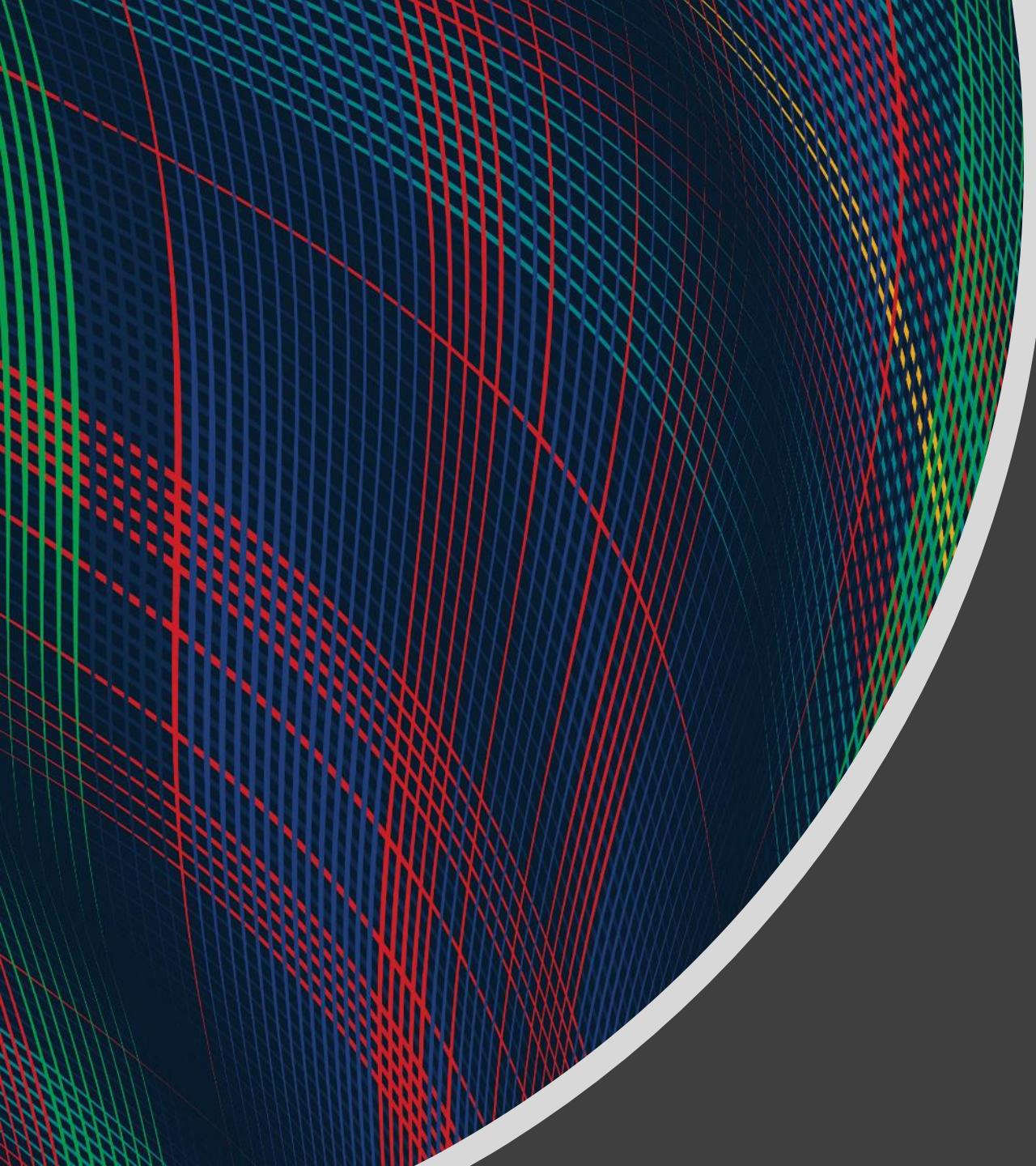
- Due Sat 10/9

Quiz3

- Mon 10/11, last 15 min. of class (probability, statistics)

Next week

- Mon 10/11: Review session
- Wed 10/13: No class
- Thu 10/14: Final exam, 1-4 pm



Mathematical Foundations for Machine Learning

Probability/Statistics
& PCA Application

Instructor: Pat Virtue

Plan

Today

Probability/Statistics

- Expectation
- Variance
- Covariance
- Covariance matrix

10-606 Application: PCA

Expectation and Variance

Expectation and Variance

The **expected value** of X is $E[X]$. Also called the mean.

- Discrete random variables:

Suppose X can take any value in the set \mathcal{X} .

$$E[X] = \sum_{x \in \mathcal{X}} xp(x)$$

- Continuous random variables:

$$E[X] = \int_{-\infty}^{+\infty} xf(x)dx$$

Expectation and Variance

The **variance** of X is $Var(X)$.

$$Var(X) = E[(X - E[X])^2]$$

- Discrete random variables:

$$Var(X) = \sum_{x \in \mathcal{X}} (x - \mu)^2 p(x)$$

$$\mu = E[X]$$

- Continuous random variables:

$$Var(X) = \int_{-\infty}^{+\infty} (x - \mu)^2 f(x) dx$$

Covariance

Expectation:

Variance:

Covariance:

Vector of random variables:

Covariance matrix:

Principal Component Analysis

Warm-up as you log in

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Dimensionality Reduction



Dimensionality Reduction



Dimensionality Reduction

For each $x^{(i)} \in \mathbb{R}^M$ find representation $z^{(i)} \in \mathbb{R}^K$ where $K \ll M$

Dimensionality Reduction

<http://timbaumann.info/svd-image-compression-demo/>

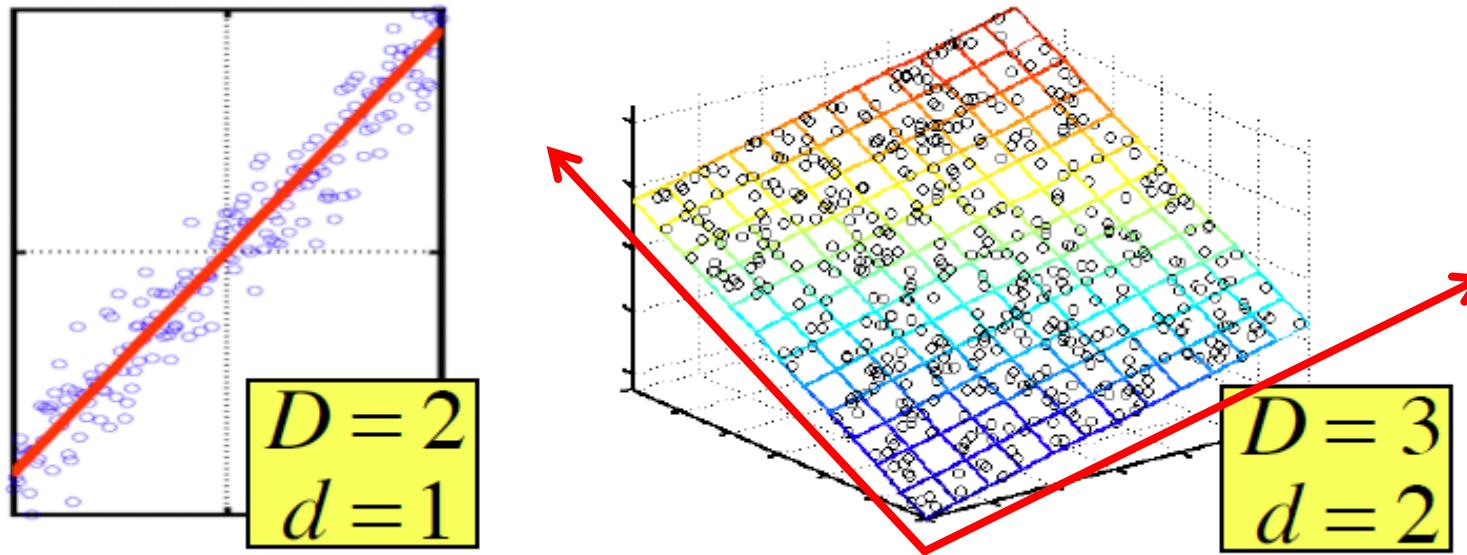
<https://cs.stanford.edu/people/karpathy/convnetjs/demo/autoencoder.html>

PCA Worksheet

Lagrange multipliers for PCA

$$\begin{aligned}\hat{\mathbf{v}} = \operatorname{argmax}_{\mathbf{v}} \quad & \mathbf{v}^T A \mathbf{v} \\ \text{s.t.} \quad & \|\mathbf{v}\|_2^2 = 1\end{aligned}$$

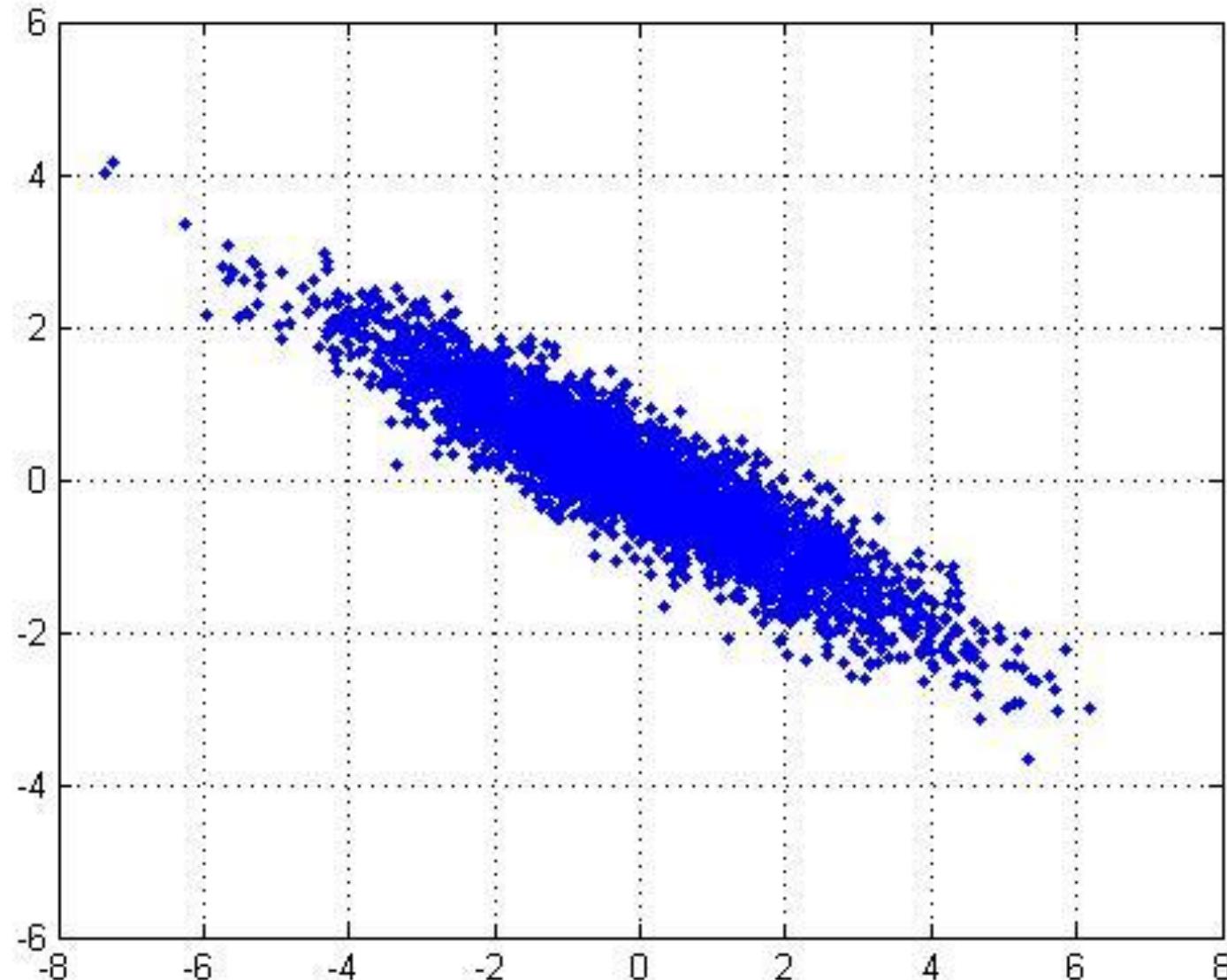
Principal Component Analysis (PCA)



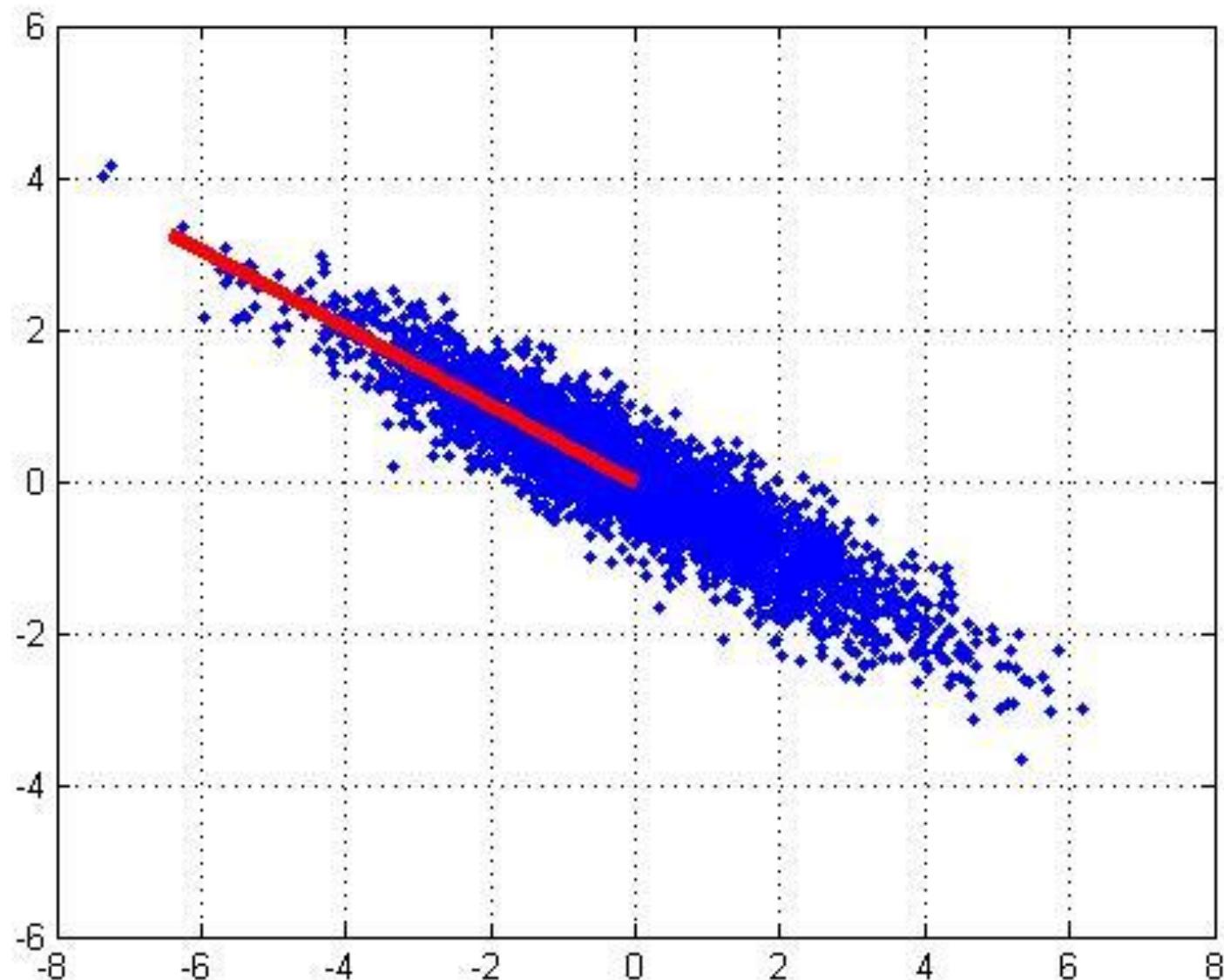
In case where data lies on or near a low d -dimensional linear subspace, axes of this subspace are an effective representation of the data.

Identifying the axes is known as [Principal Components Analysis](#), and can be obtained by using classic matrix computation tools (Eigen or Singular Value Decomposition).

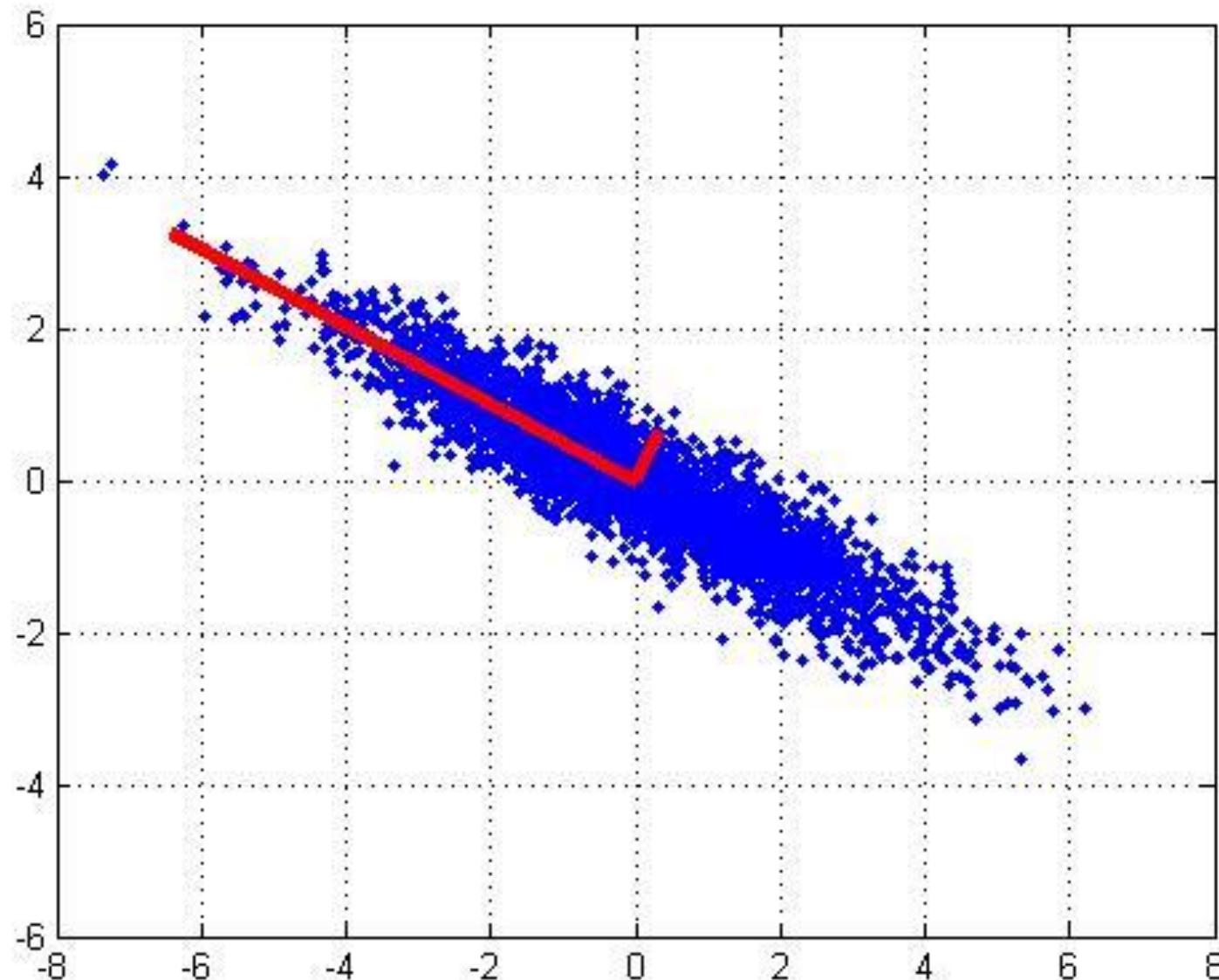
2D Gaussian dataset



1st PCA axis



2nd PCA axis



Data for PCA

$$\mathcal{D} = \{\mathbf{x}^{(i)}\}_{i=1}^N$$

$$\mathbf{X} = \begin{bmatrix} (\mathbf{x}^{(1)})^T \\ (\mathbf{x}^{(2)})^T \\ \vdots \\ (\mathbf{x}^{(N)})^T \end{bmatrix}$$

We assume the data is **centered**

$$\mu = \frac{1}{N} \sum_{i=1}^N \mathbf{x}^{(i)} = \mathbf{0}$$

Q: What if
your data is
not centered?

A: Subtract
off the
sample mean

Sample Covariance Matrix

The sample covariance matrix is given by:

$$\Sigma_{jk} = \frac{1}{N} \sum_{i=1}^N (x_j^{(i)} - \mu_j)(x_k^{(i)} - \mu_k)$$

Since the data matrix is centered, we rewrite as:

$$\Sigma = \frac{1}{N} \mathbf{X}^T \mathbf{X}$$

$$\mathbf{X} = \begin{bmatrix} (\mathbf{x}^{(1)})^T \\ (\mathbf{x}^{(2)})^T \\ \vdots \\ (\mathbf{x}^{(N)})^T \end{bmatrix}$$

PCA Algorithm

Input: $\mathbf{X}, \mathbf{X}_{test}, K$

1. Center data (and scale each axis) based on training data $\rightarrow \mathbf{X}, \mathbf{X}_{test}$
2. $\mathbf{V} = \text{eigenvectors}(\mathbf{X}^T \mathbf{X})$
3. Keep only the top K eigenvectors: \mathbf{V}_K
4. $\mathbf{Z}_{test} = \mathbf{X}_{test} \mathbf{V}_K$

Optionally, use \mathbf{V}_K^T to rotate \mathbf{Z}_{test} back to original subspace \mathbf{X}'_{test} and uncenter