Naïve Bayes Classifier

Machine Learning 10-601B

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Let’s learn classifiers by learning P(Y|X)

Consider Y=Wealth, X=<Gender, HoursWorked>

gender	hours_worked	wealth	
Female	v0:40.5-	poor	0.253122
		rich	0.0245895
	v1:40.5+	poor	0.0421768
		rich	0.0116293
Male	v0:40.5-	poor	0.331313
		rich	0.0971295
	v1:40.5+	poor	0.134106
		rich	0.105933

P(gender, hours_worked, wealth)
=> P(wealth| gender, hours_worked)

Gender	HrsWorked	P(rich | G,HW)	P(poor | G,HW)
F	<40.5	.09	.91
F	>40.5	.21	.79
M	<40.5	.23	.77
M	>40.5	.38	.62

How many parameters must we estimate?

Gender	HrsWorked	P(rich | G,HW)	P(poor | G,HW)
F	<40.5	.09	.91
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Suppose $X = \langle X_1, \ldots X_n \rangle$
where X_i and Y are boolean RV's

To estimate $P(Y | X_1, X_2, \ldots X_n)$

2^n quantities need to be estimated!

If we have 30 boolean X_i's: $P(Y | X_1, X_2, \ldots X_{30})$

$2^{30} \sim 1$ billion!

You need lots of data or a very small *n*

Can we reduce params using Bayes Rule?

Suppose $X = \langle X_1, \ldots X_n \rangle$
where X_i and Y are boolean RV's

$$P(Y|X) = \frac{P(X|Y)P(Y)}{P(X)}$$

How many parameters for $P(X|Y) = P(X_1, \ldots X_n | Y)$?

$(2^n-1) \times 2$

How many parameters for $P(Y)$?

1

Naïve Bayes

Naïve Bayes assumes

$$P(X_1 \ldots X_n|Y) = \prod_i P(X_i|Y)$$

i.e., that X_i and X_j are conditionally independent given Y, for all $i \neq j$

Conditional independence

- Two variables A,B are *independent* if

$$P(A \wedge B) = P(A) * P(B)$$

$$\forall a, b : P(A = a \wedge B = b) = P(A = a) * P(B = b)$$

- Two variables A,B are *conditionally independent given* C if

$$P(A, B \mid C) = P(A \mid C) * P(B \mid C)$$

$$\forall a, bc : P(A = a \wedge B = b \mid C = c) = P(A = a \mid C = c) * P(B = b \mid C = c)$$

Conditional Independence

Definition: X is conditionally independent of Y given Z, if the probability distribution governing X is independent of the value of Y, given the value of Z

$$(\forall i, j, k) P(X = x_i | Y = y_j, Z = z_k) = P(X = x_i | Z = z_k)$$

Which we often write

$$P(X|Y, Z) = P(X|Z)$$

E.g. $P(Thunder|Rain, Lightning) = P(Thunder|Lightning)$

Naïve Bayes uses assumption that the X_i are conditionally independent, given Y

Given this assumption, then:

$$P(X_1, X_2|Y) = P(X_1|X_2, Y)P(X_2|Y)$$

$$= P(X_1|Y)P(X_2|Y)$$

Chain rule

Conditional Independence

in general: $P(X_1...X_n|Y) = \prod_i P(X_i|Y)$

(2^n-1)x2

2n

Reducing the number of parameters to estimate

$$P(Y|X_1,\ldots,X_n) = \frac{P(X_{1,}\ldots,X_n|Y)P(Y)}{P(X_1,\ldots,X_n)}$$

To make this tractable we naively assume conditional independence of the features given the class: ie

$$P(X_1\ldots,X_n|Y) = P(X_1|Y)\cdot P(X_2|Y)\cdots P(X_n|Y)$$

Now: I only need to estimate … parameters:

$$P(X_1|Y), P(X_2|Y), \cdots, P(X_n|Y), P(Y)$$

How many parameters to describe $P(X_1\ldots X_n|Y)$? $P(Y)$?

- Without conditional indep assumption? $(2^n-1)\times 2+1$
- With conditional indep assumption? $2n+1$

Naïve Bayes Algorithm – discrete X_i

- Train Naïve Bayes (given data for X and Y)

  for each* value y_k

  estimate $\pi_k \equiv P(Y = y_k)$

  for each* value x_{ij} of each attribute X_i

  estimate $\theta_{ijk} \equiv P(X_i = x_{ij} | Y = y_k)$

Training Naïve Bayes Classifier Using MLE

- From the data D, estimate *class priors*.
  - For each possible value of Y, estimate $Pr(Y=y_1), Pr(Y=y_2), \ldots Pr(Y=y_k)$
  - An MLE estimate: $\widehat{\pi}_k = \widehat{P}(Y = y_k) = \frac{\#D\{Y = y_k\}}{|D|}$
- From the data, estimate the conditional probabilities
  - If every X_i has values $x_{i1},\ldots,x_{ik}$
    - for each y_i and each X_i estimate $q(i,j,k)=Pr(X_i=x_{ij} | Y=y_i)$

$$\widehat{\theta}_{ijk} = \widehat{P}(X_i = x_{ij} | Y = y_k) = \frac{\#D\{X_i = x_{ij} \wedge Y = y_k\}}{\#D\{Y = y_k\}}$$

Number of items in dataset D for which $Y=y_k$

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  estimate $\theta_{ijk} \equiv P(X_i = x_{ij} | Y = y_k)$

- Classify (X^{new})

$$Y^{new} \leftarrow \arg\max_{y_k} \; P(Y = y_k) \prod_i P(X_i^{new} | Y = y_k)$$

$$Y^{new} \leftarrow \arg\max_{y_k} \; \pi_k \prod_i \theta_{ijk}$$

* probabilities must sum to 1, so need estimate only n-1 of these...

Example: Live in Sq Hill? P(S|G,D,E)

- S=1 iff live in Squirrel Hill
- G=1 iff shop at SH Giant Eagle
- D=1 iff Drive or Carpool to CMU
- E=1 iff Even # letters last name

What probability parameters must we estimate?

P(S=1) :	P(S=0) :
P(D=1 | S=1) :	P(D=0 | S=1) :
P(D=1 | S=0) :	P(D=0 | S=0) :
P(G=1 | S=1) :	P(G=0 | S=1) :
P(G=1 | S=0) :	P(G=0 | S=0) :
P(E=1 | S=1) :	P(E=0 | S=1) :
P(E=1 | S=0) :	P(E=0 | S=0) :

Naïve Bayes: Subtlety #1

If unlucky, our MLE estimate for $P(X_i \mid Y)$ might be zero. (e.g., nobody in your sample has X_i = <40.5 and Y=rich)

- Why worry about just one parameter out of many?

$$P(X_1 \ldots X_n|Y) = \prod_i \underbrace{P(X_i|Y)}_{\text{If one of these terms is 0...}}$$

- What can be done to avoid this?

Estimating Parameters

- Maximum Likelihood Estimate (MLE): choose θ that maximizes probability of observed data $\mathcal{D}$

$$\widehat{\theta} \;=\; \arg\max_{\theta} \quad P(\mathcal{D} \mid \theta)$$

- Maximum a Posteriori (MAP) estimate: choose θ that is most probable given prior probability and the data

$$\begin{aligned}\widehat{\theta} \;&=\; \arg\max_{\theta} \quad P(\theta \mid \mathcal{D}) \\ &= \arg\max_{\theta} \;=\; \frac{P(\mathcal{D} \mid \theta) P(\theta)}{P(\mathcal{D})}\end{aligned}$$

Estimating Parameters: Y, X_i discrete-valued

Maximum likelihood estimates:

$$\widehat{\pi}_k = \widehat{P}(Y = y_k) = \frac{\#D\{Y = y_k\}}{|D|}$$

$$\hat{\theta}_{ijk} = \hat{P}(X_i = x_j | Y = y_k) = \frac{\#D\{X_i = x_j \wedge Y = y_k\}}{\#D\{Y = y_k\}}$$

MAP estimates (Beta, Dirichlet priors):

$$\hat{\pi}_k = \hat{P}(Y = y_k) = \frac{\#D\{Y = y_k\} + (\beta_k - 1)}{|D| + \sum_m (\beta_m - 1)}$$

$$\hat{\theta}_{ijk} = \hat{P}(X_i = x_j | Y = y_k) = \frac{\#D\{X_i = x_j \wedge Y = y_k\} + (\beta_k - 1)}{\#D\{Y = y_k\} + \sum_m (\beta_m - 1)}$$

Only difference: "imaginary" examples

Naïve Bayes: Subtlety #2

Often the X_i are not really conditionally independent

- We use Naïve Bayes in many cases anyway, and it often works pretty well
  - often the right classification, even when not the right probability (see [Domingos&Pazzani, 1996])
- What is effect on estimated P(Y|X)?
  - Special case: what if we add two copies: $X_i = X_k$

Special case: what if we add two copies: $X_i = X_k$

$$P(X_1 \ldots X_n|Y) = \prod_i \underbrace{P(X_i|Y)}_{\text{Redundant terms}}$$

About Naïve Bayes

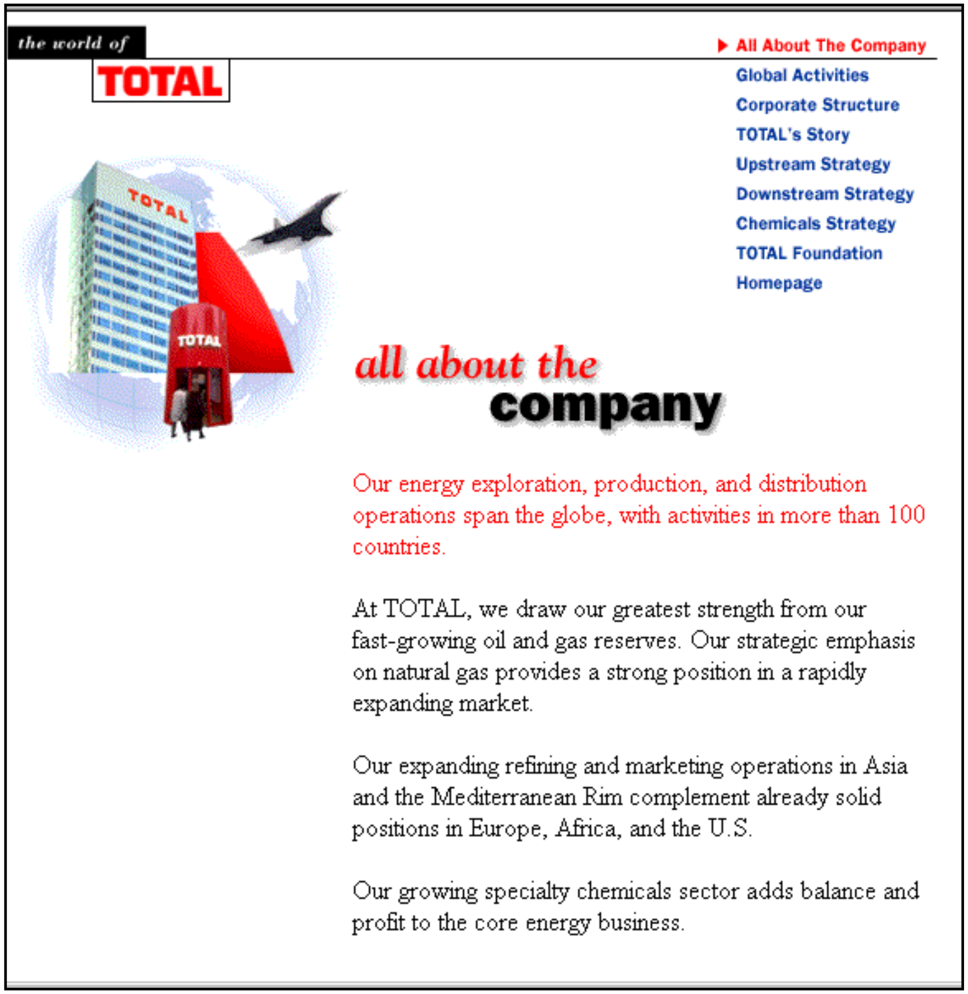
- Naïve Bayes is blazingly fast and quite robust!

Learning to classify text documents

- Classify which emails are spam?
- Classify which emails promise an attachment?
- Classify which web pages are student home pages?

- How shall we represent text documents for Naïve Bayes?

Baseline: Bag of Words Approach



aardvark	0
about	2
all	2
Africa	1
apple	0
anxious	0
...	
gas	1
...	
oil	1
…	
Zaire	0

Learning to classify document: P(Y|X) the "Bag of Words" model

- Y discrete valued. e.g., Spam or not
- $X = <X_1, X_2, \ldots X_n>$ = document

- X_i is a random variable describing the word at position i in the document
- possible values for X_i : any word w_k in English

- Document = bag of words: the vector of counts for all w_k's
  - (like #heads, #tails, but we have more than 2 values)

Naïve Bayes Algorithm – discrete X_i

- Train Naïve Bayes (examples)

  for each value y_k

  estimate $\pi_k \equiv P(Y = y_k)$

  for each value x_j of each attribute X_i

  estimate $\theta_{ijk} \equiv P(X_i = x_j | Y = y_k)$

  prob that word x_j appears in position i, given $Y=y_k$

- Classify (X^{new})

$$Y^{new} \leftarrow \arg\max_{y_k} \; P(Y = y_k) \prod_i P(X_i^{new} | Y = y_k)$$

$$Y^{new} \leftarrow \arg\max_{y_k} \; \pi_k \prod_i \theta_{ijk}$$

* Additional assumption: word probabilities are position independent

$$\theta_{ijk} = \theta_{mjk} \quad \text{for all } i, m$$

MAP estimates for bag of words

MAP estimate for multinomial

$$\theta_i = \frac{\alpha_i + \beta_i - 1}{\sum_{m=1}^{k} \alpha_m + \sum_{m=1}^{k} (\beta_m - 1)}$$

$$\theta_{aardvark} = P(X_i = \text{aardvark}) = \frac{\#\text{ observed 'aardvark' } + \#\text{ hallucinated 'aardvark'} - 1}{\#\text{ observed words } + \#\text{ hallucinated words} - k}$$

What β’ s should we choose?

Twenty NewsGroups

Given 1000 training documents from each group
Learn to classify new documents according to which newsgroup it came from

comp.graphics	misc.forsale
comp.os.ms-windows.misc	rec.autos
comp.sys.ibm.pc.hardware	rec.motorcycles
comp.sys.mac.hardware	rec.sport.baseball
comp.windows.x	rec.sport.hockey
alt.atheism	sci.space
soc.religion.christian	sci.crypt
talk.religion.misc	sci.electronics
talk.politics.mideast	sci.med
talk.politics.misc	
talk.politics.guns	

Naive Bayes: 89% classification accuracy

What you should know:

- Training and using classifiers based on Bayes rule
- Conditional independence
  - What it is
  - Why it’s important
- Naïve Bayes
  - What it is
  - Why we use it so much
  - Training using MLE, MAP estimates