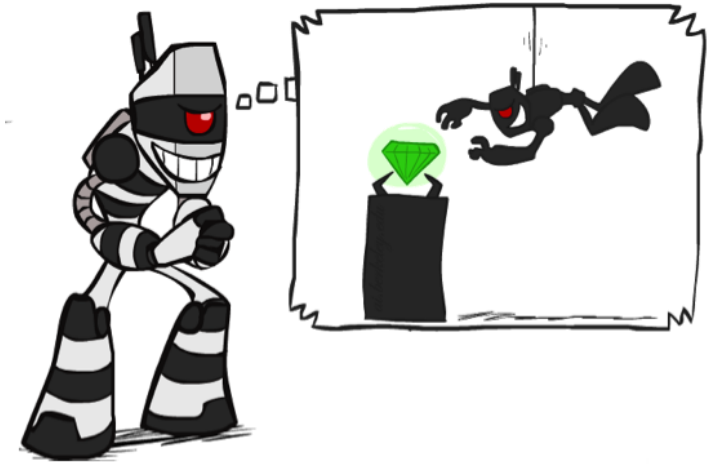
07-280 AI & ML I
Constraint Satisfaction Problems (CSPs)

Instructors:

Pat Virtue & Nihar Shah

What is Search For?

- Planning: sequences of actions
  - The path to the goal is the important thing
  - Paths have various costs, depths
  - Heuristics give problem-specific guidance



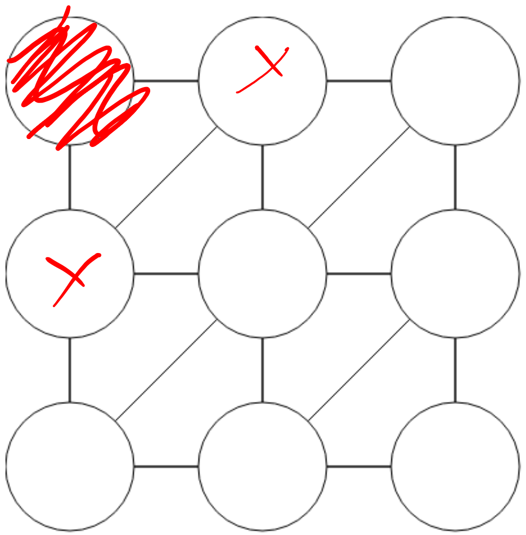
- Identification: assignments to variables
  - The goal itself is important, not the path
  - All paths at the same depth (for some formulations)

Are the warm-up assignments
planning or identification problems?

Slide/Image credit: Ketrina Yim, UCB CS 188

Warm-up as You Walk In

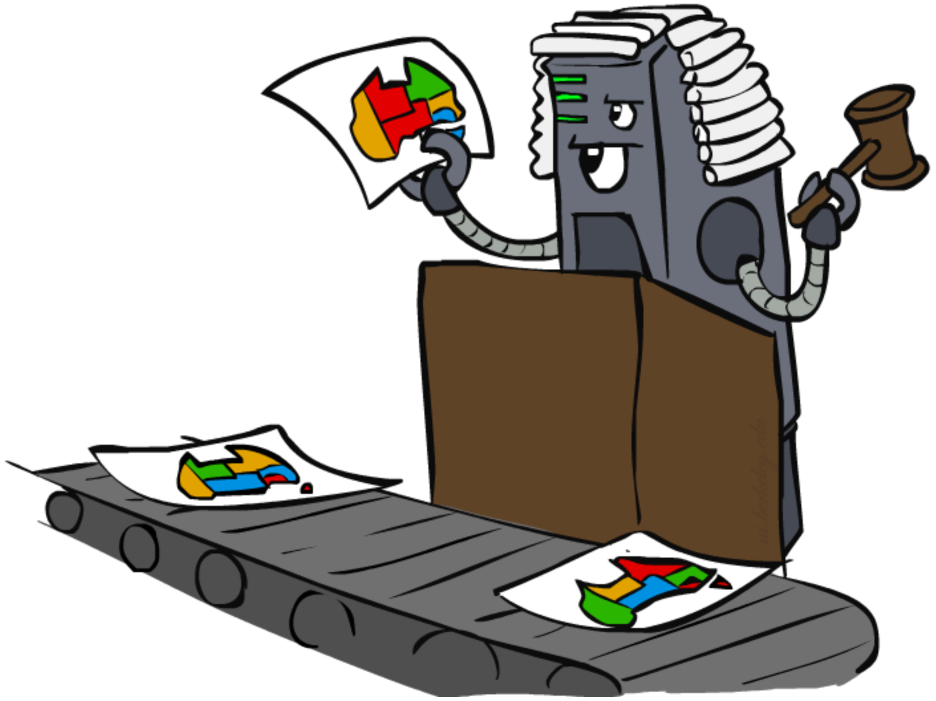
Assign Red, Green, or Blue
Neighbors must be different



Sudoku

1	3	4	2
2	2	1	
3		3	
4			4

Constraint Satisfaction Problems



CSP is a special class of search problems

- Mostly identification problems
- Have specialized algorithms for them

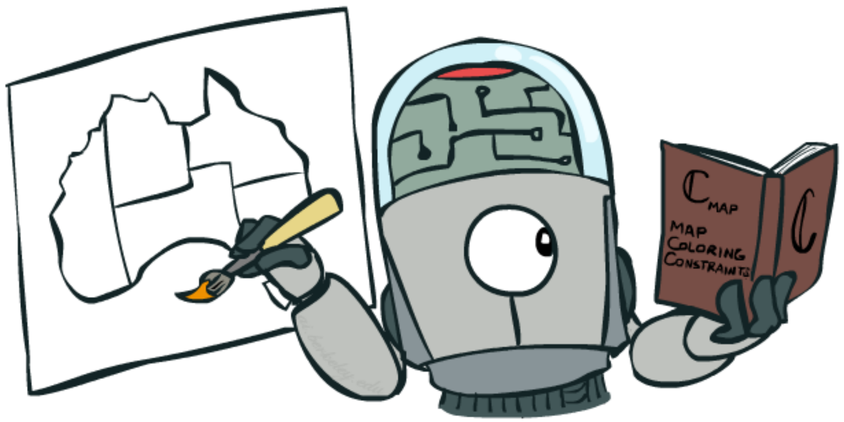
Standard search problems:

- State is an arbitrary data structure
- Goal test can be any function over states

Constraint satisfaction problems (CSPs):

n
- State is defined by variables X_i with values from a domain D (sometimes D depends on i)

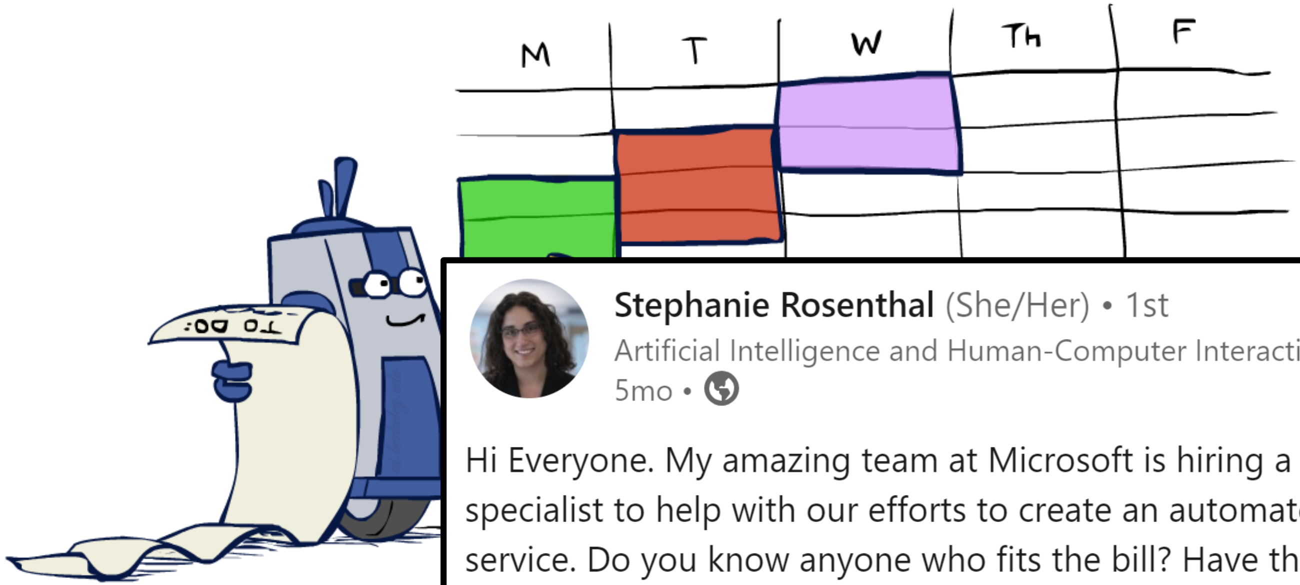
d
- Goal test is a set of constraints specifying allowable combinations of values for subsets of variables



Why study CSPs?

Many real-world problems can be formulated as CSPs

- Assignment problems: e.g., who teaches what class
- Timetabling problems: e.g., which class is offered when and where?
- Hardware configuration
- Transportation scheduling
- Factory scheduling
- Circuit layout
- Fault diagnosis
- … lots more!



- Sometimes involve real-valued variables…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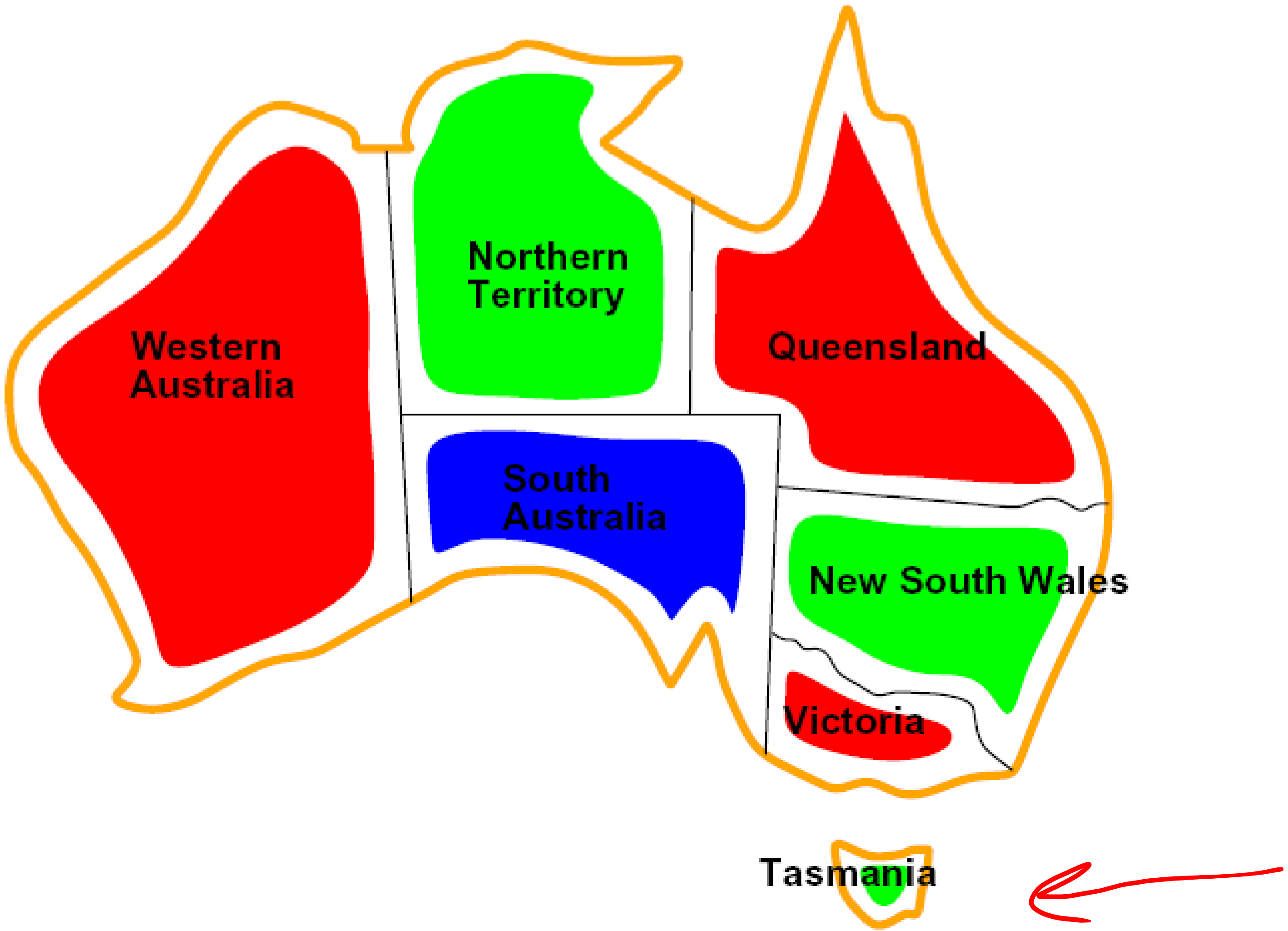
CSP Examples

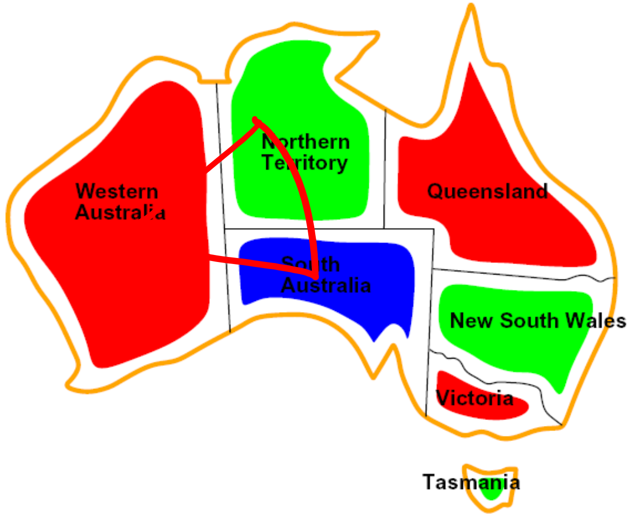


Example: Map Coloring

Alldiff(WA, NT, SA)

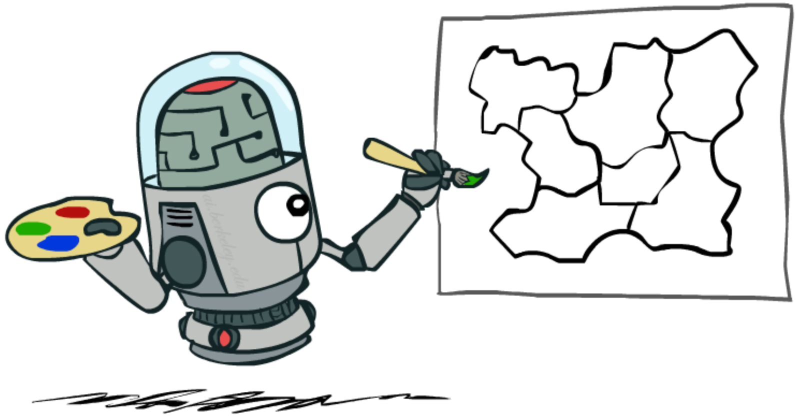
- Variables: WA, NT, Q, NSW, V, SA, T
- Domains: $D = \{\text{red}, \text{green}, \text{blue}\}$
- Constraints: adjacent regions must have different colors

$$\mathrm{WA} \neq \mathrm{NT}$$



- Solutions are assignments satisfying all constraints, e.g.:

{WA=red, NT=green, Q=red, NSW=green, V=red, SA=blue, T=green}



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Example: Map Coloring

Google OR Tools

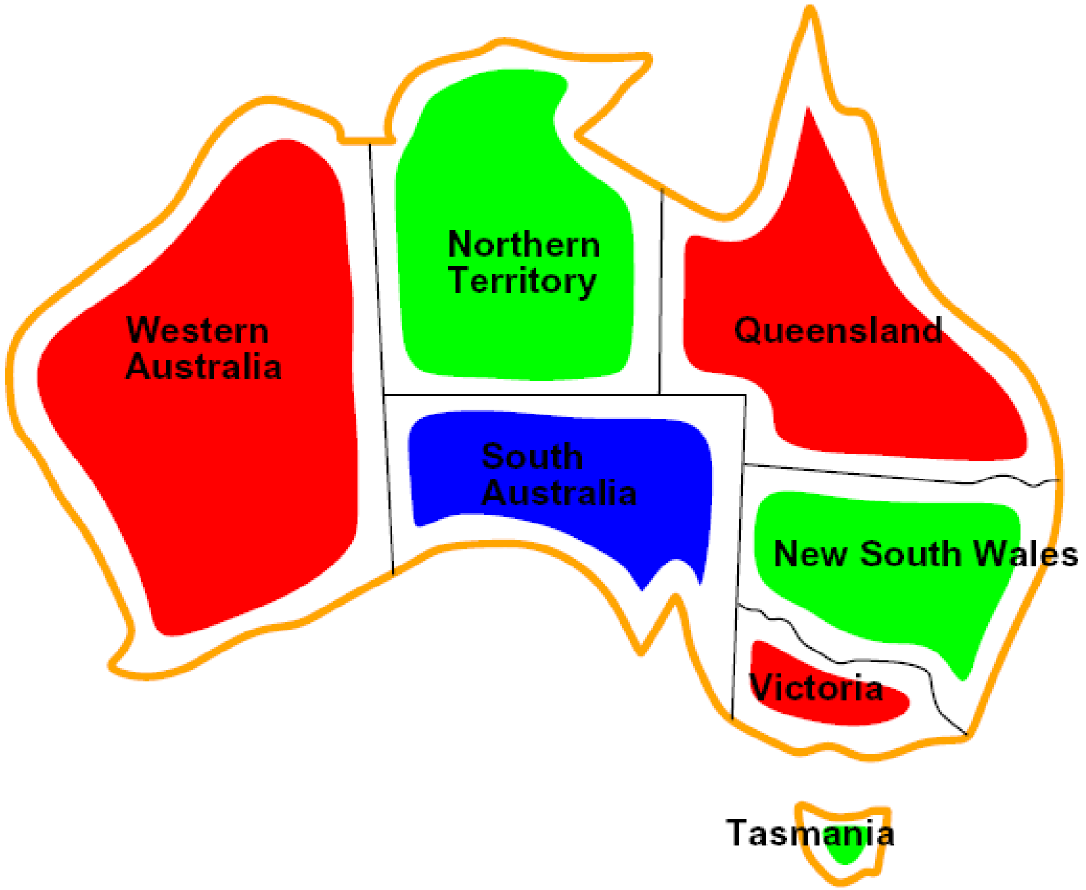
```python
from ortools.sat.python import cp_model

model = cp_model.CpModel()


# All CSP variables:
# Domain: {0,1,2} representing three colors
variables = {
    "WA":  model.NewIntVar(0, 2, "WA"),
    "NT":  model.NewIntVar(0, 2, "NT"),
    "SA":  model.NewIntVar(0, 2, "SA"),
    "Q":   model.NewIntVar(0, 2, "Q"),
    "NSW": model.NewIntVar(0, 2, "NSW"),
    "V":   model.NewIntVar(0, 2, "V"),
    "T":   model.NewIntVar(0, 2, "T"),
}
```

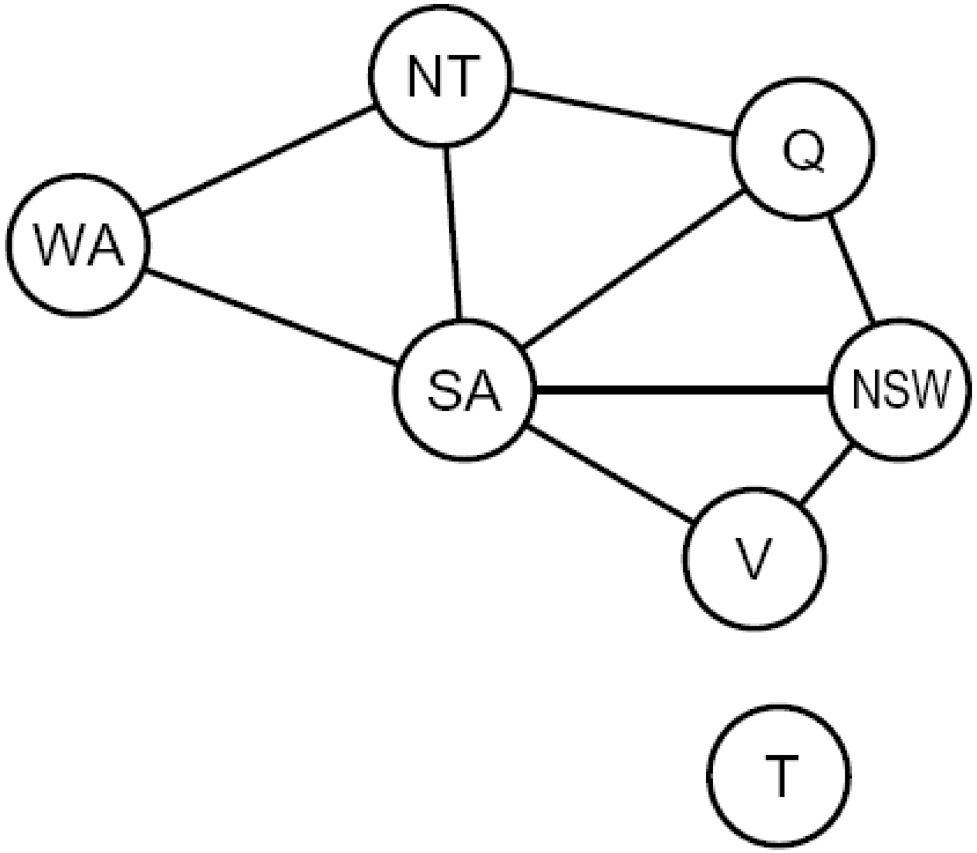
```python
# Edges (related to binary constraints)
constraints = [
    ("WA", "NT"),
    ("WA", "SA"),
    ("NT", "SA"),
    ("NT", "Q"),
    ("SA", "Q"),
    ("SA", "NSW"),
    ("SA", "V"),
    ("Q", "NSW"),
    ("NSW", "V"),
]


# Add binary inequality constraints:
# adjacent regions must differ in color
for v1, v2 in constraints:
    model.Add(variables[v1] != variables[v2])


# Solve
solver = cp_model.CpSolver()
status = solver.Solve(model)
```

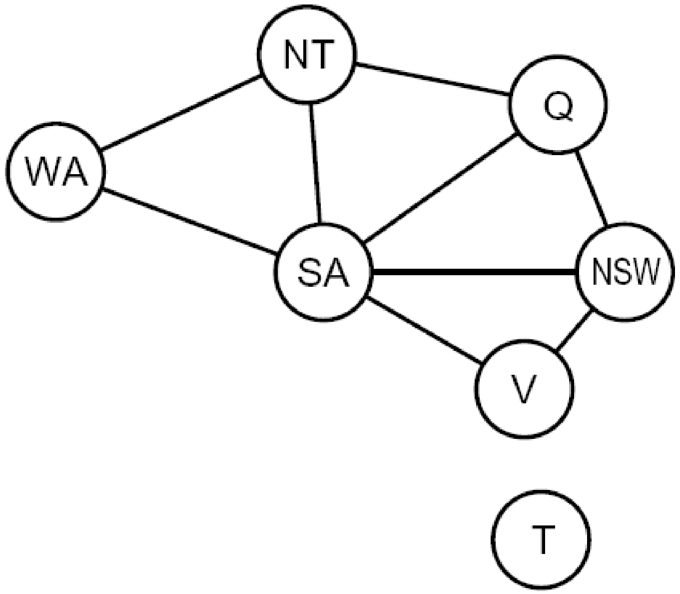
Constraint Graphs





Constraint Graphs

- Binary CSP: each constraint relates (at most) two variables
- Binary constraint graph: nodes are variables, arcs show constraints
- General-purpose CSP algorithms use the graph structure to speed up search. E.g., Tasmania is an independent subproblem!



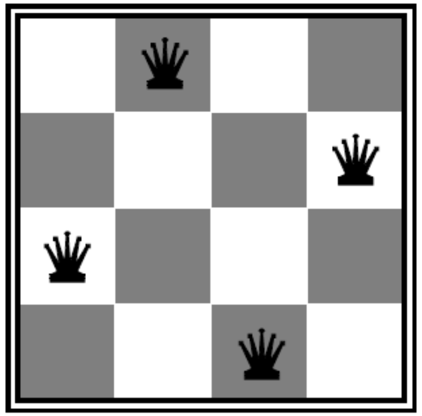
Varieties of CSPs and Constraints

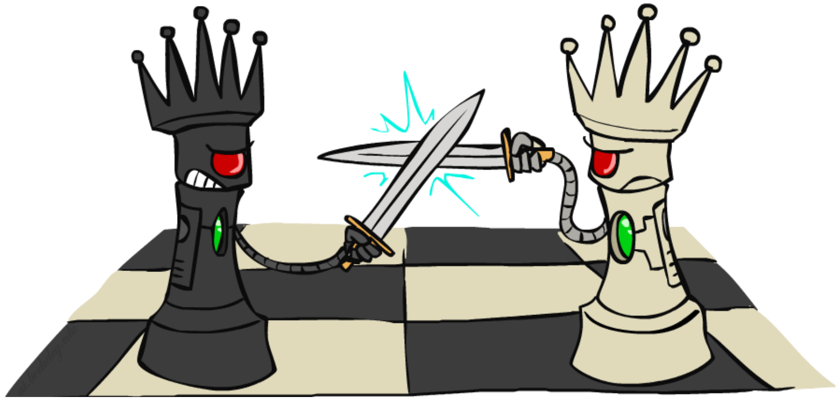
Slide/Image credit: Ketrina Yim, UCB CS 188

Example: N-Queens

$N = 4$

- Formulation 1:
  - Variables: X_{ij}
  - Domains: $\{0, 1\}$
  - Constraints



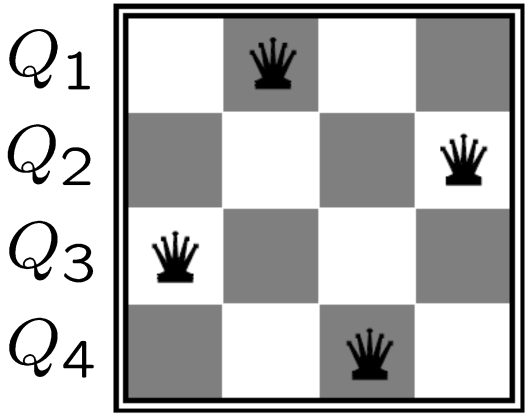


$$\forall i, j, k \ \ (X_{ij}, X_{ik}) \in \{(0,0), (0,1), (1,0)\}$$
$$\forall i, j, k \ \ (X_{ij}, X_{kj}) \in \{(0,0), (0,1), (1,0)\}$$
$$\forall i, j, k \ \ (X_{ij}, X_{i+k,j+k}) \in \{(0,0), (0,1), (1,0)\}$$
$$\forall i, j, k \ \ (X_{ij}, X_{i+k,j-k}) \in \{(0,0), (0,1), (1,0)\}$$

$$\sum_{i,j} X_{ij} = N$$

Example: N-Queens

- Formulation 2:
  - Variables: Q_k
  - Domains: $\{1, 2, 3, \ldots N\}$
  - Constraints:



Implicit: $\forall i, j$ non-threatening(Q_i, Q_j)

Explicit: $(Q_1, Q_2) \in \{(1,3), (1,4), \ldots\}$

$\cdots$

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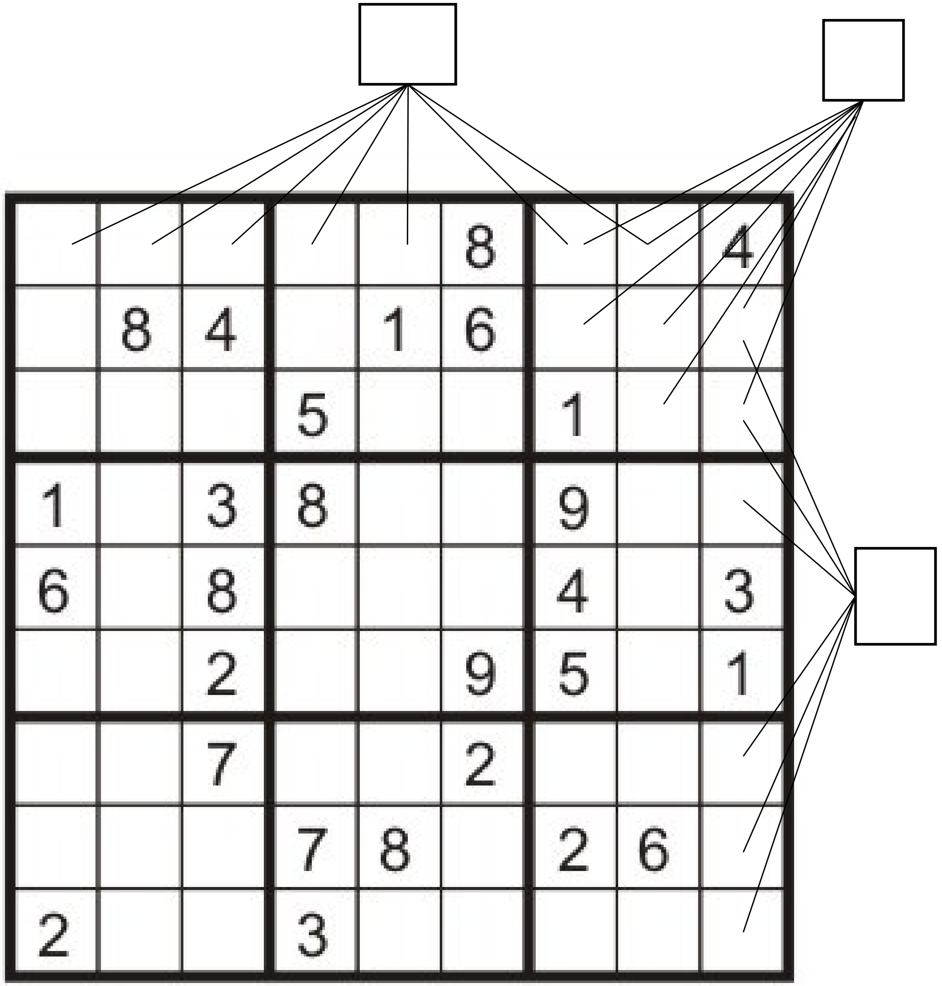
Example: Sudoku



- Variables: Each (open) square
- Domains: {1,2,…,9}
- Constraints:

  9-way alldiff for each column

  9-way alldiff for each row

  9-way alldiff for each region

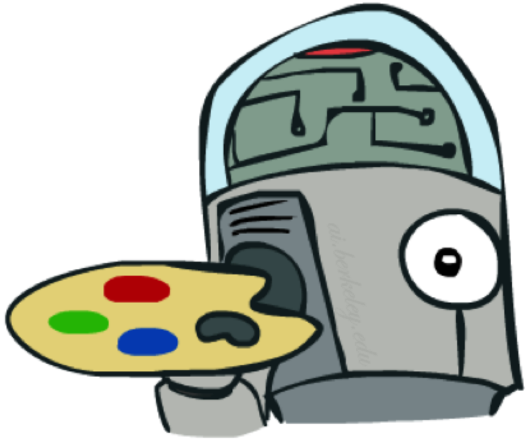
  (or can have a bunch of pairwise inequality constraints)

Varieties of CSPs

- Discrete Variables

We will cover today

  - Finite domains
    - Size d means $O(d^n)$ complete assignments
    - E.g., Boolean CSPs, including Boolean satisfiability (NP-complete)



  - Infinite domains (integers, strings, etc.)
    - E.g., job scheduling, variables are start/end times for each job
    - Linear constraints solvable, nonlinear undecidable

We will cover in a 07-380

- Continuous variables
  - E.g., start/end times for Hubble Telescope observations
  - Linear constraints solvable in polynomial time

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Varieties of Constraints

- Varieties of Constraints
  - Unary constraints involve a single variable (equivalent to reducing domains), e.g.:
    $SA \neq green$
    Focus of today
  - Binary constraints involve pairs of variables, e.g.:
    $SA \neq WA$
  - Higher-order constraints involve 3 or more variables:
    e.g., sudoku constraints

- Preferences (soft constraints):
  - E.g., red is better than green
  - Often representable by a cost for each variable assignment
  - Gives constrained optimization problems

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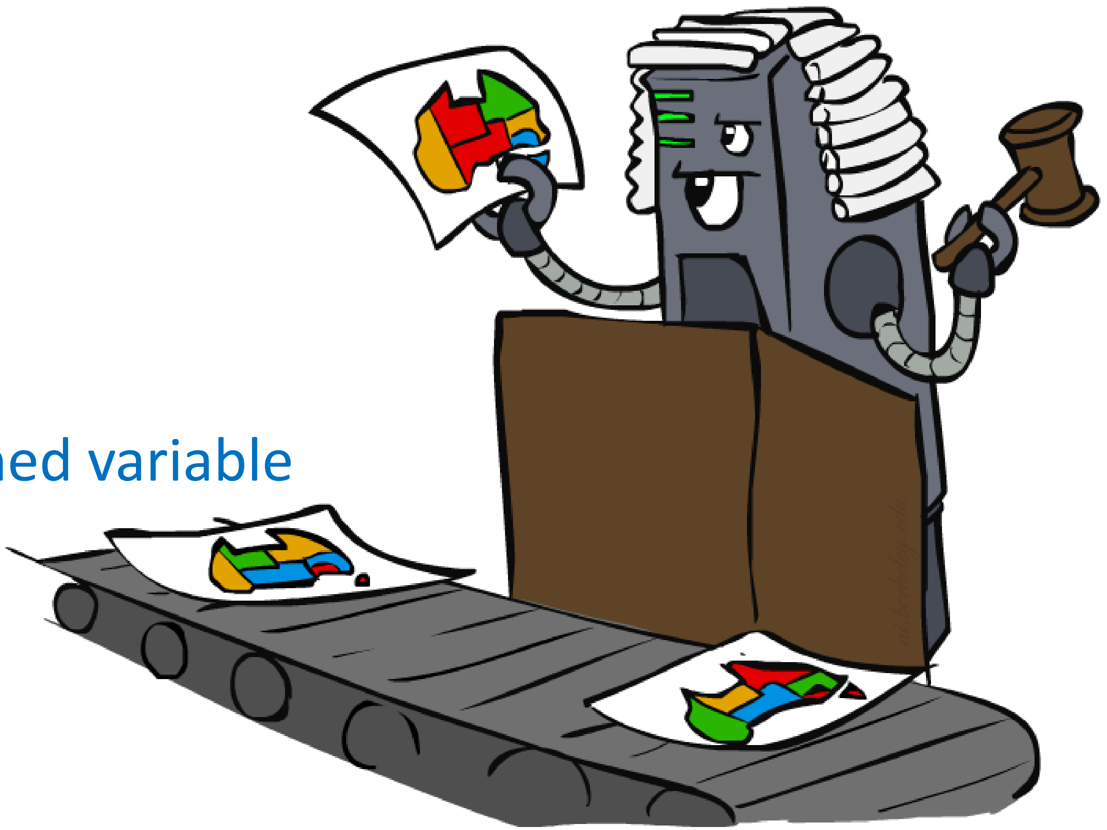
Solving CSPs

Standard Search Formulation

- Standard search formulation of CSPs

- States defined by the values assigned so far (partial assignments)
  - Initial state: the empty assignment, {}
  - Successor function: assign a value to an $\boxed{\text{unassigned variable}}$ →Can be any unassigned variable
  - Goal test: the current assignment is complete and satisfies all constraints

- We'll start with the straightforward, naïve approach, then improve it



Poll 1: Search for CSPs

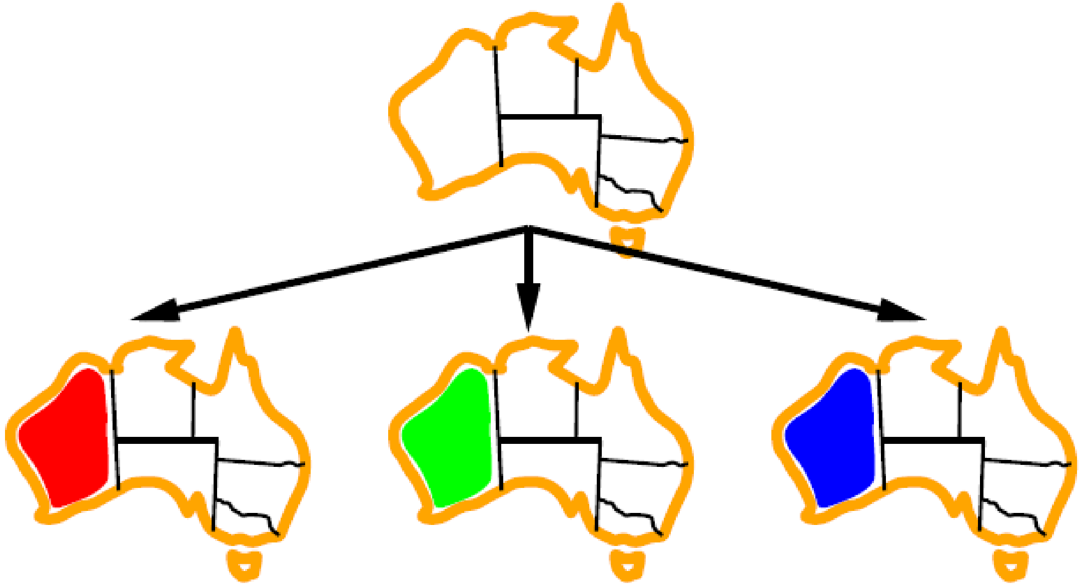
Should we use BFS or DFS?

A

B : BFS

C

D: DFS



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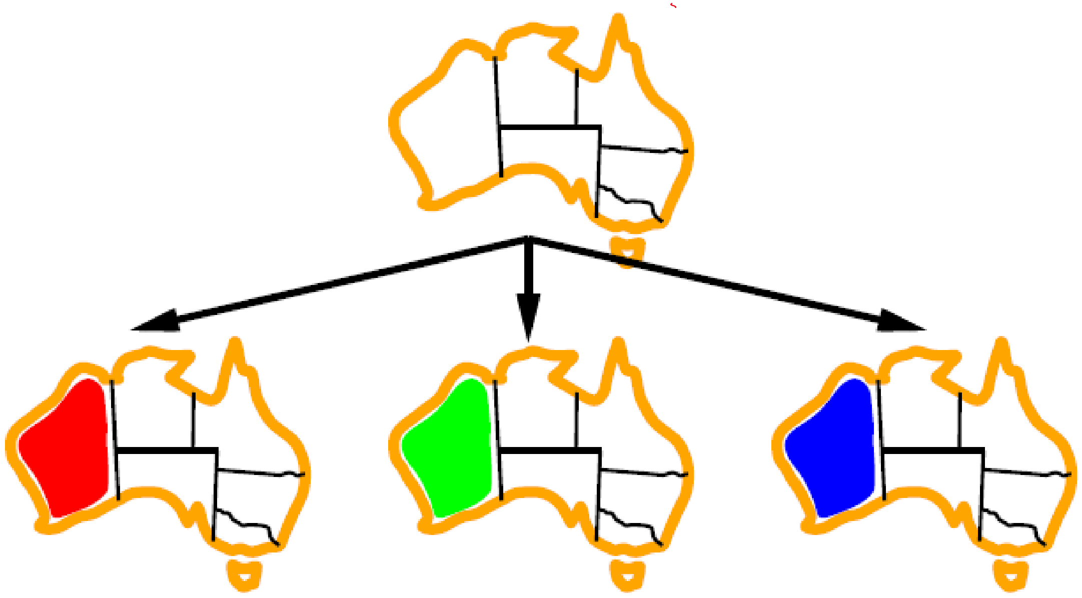
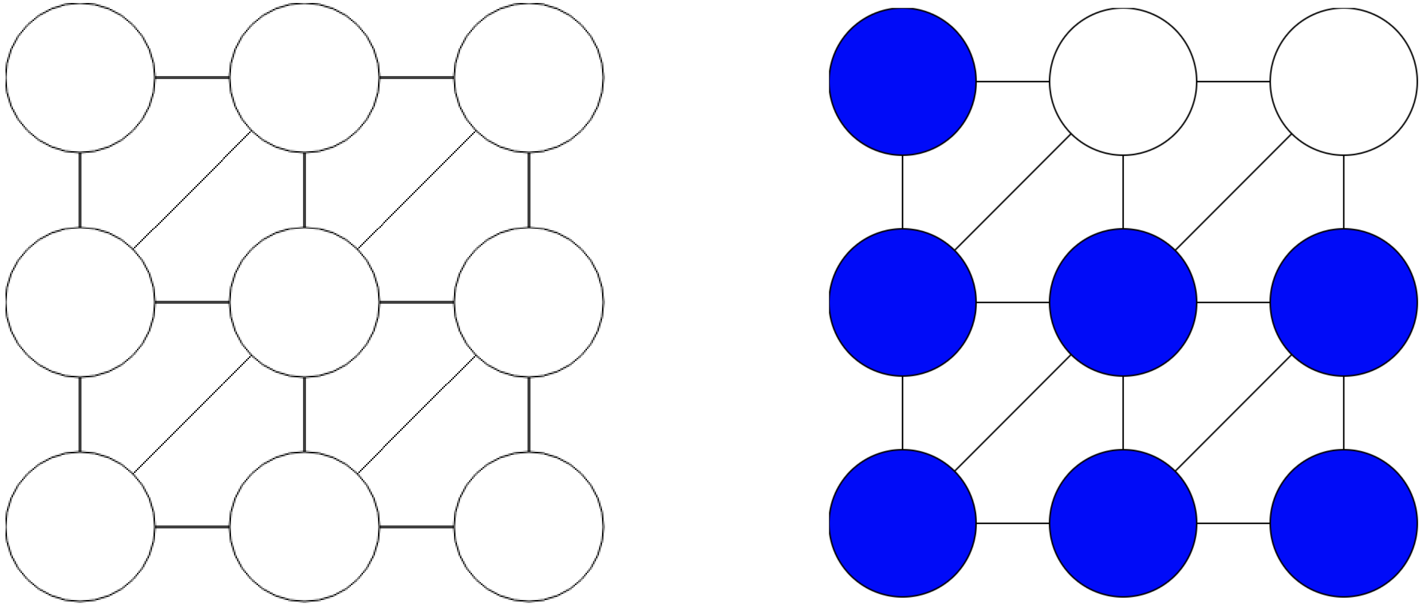
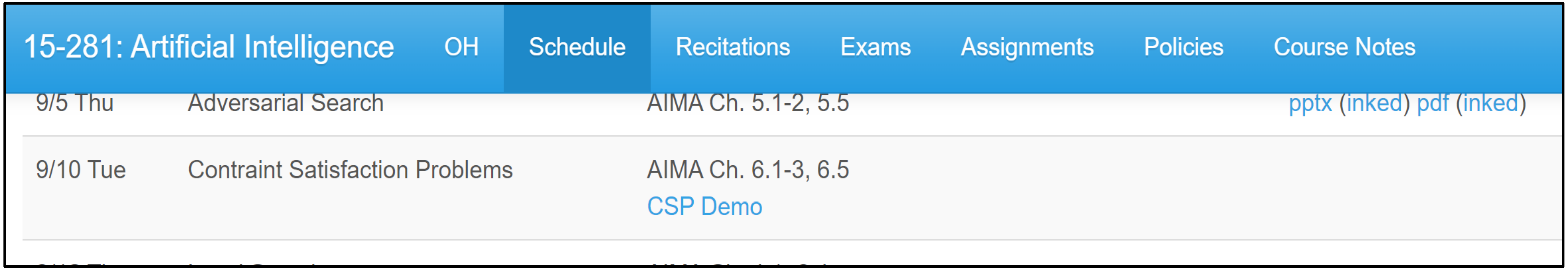
Depth First Search

- At each node, assign a value from the domain to the variable
- Check feasibility (constraints) when the assignment is complete



Demo – Naïve Search





15-281: Artificial Intelligence | OH | Schedule | Recitations | Exams | Assignments | Policies | Course Notes

9/5 Thu	Adversarial Search	AIMA Ch. 5.1-2, 5.5	pptx (inked) pdf (inked)
9/10 Tue	Contraint Satisfaction Problems	AIMA Ch. 6.1-3, 6.5 CSP Demo	

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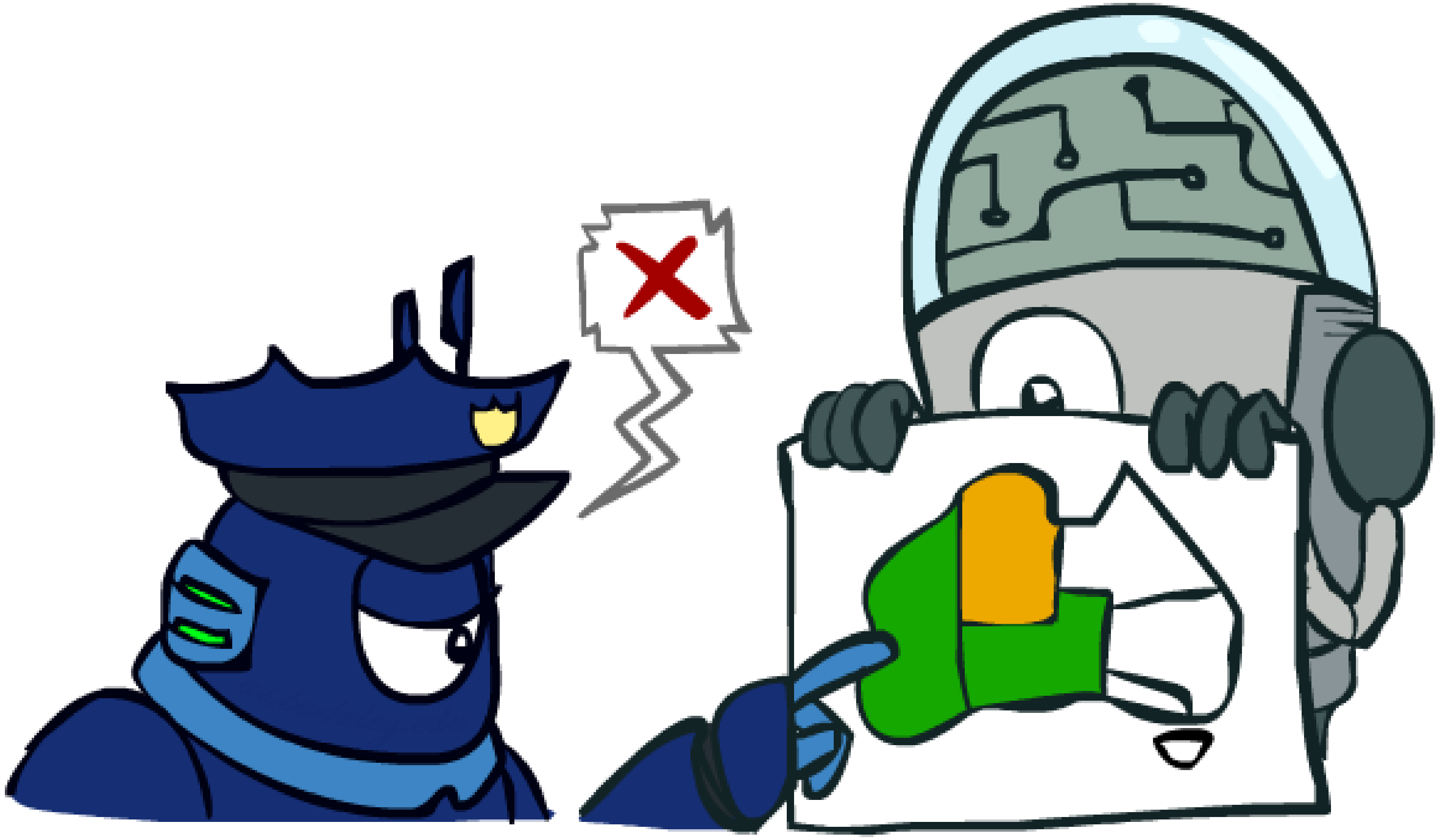
Backtracking Search



Backtracking Search

Backtracking search is the basic uninformed algorithm for solving CSPs

Backtracking search = DFS + two improvements

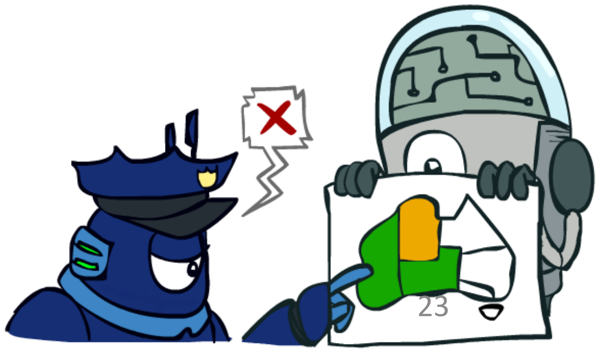
Idea 1: One variable at a time

- Variable assignments are commutative
  - [WA = red then NT = green] same as [NT = green then WA = red]
- Only need to consider assign value to a single variable at each step

Idea 2: Check constraints as you go

- Consider only values which do not conflict previous assignments
- May need some computation to check the constraints
- “Incremental goal test”

Can solve n-queens for $n \approx 25$



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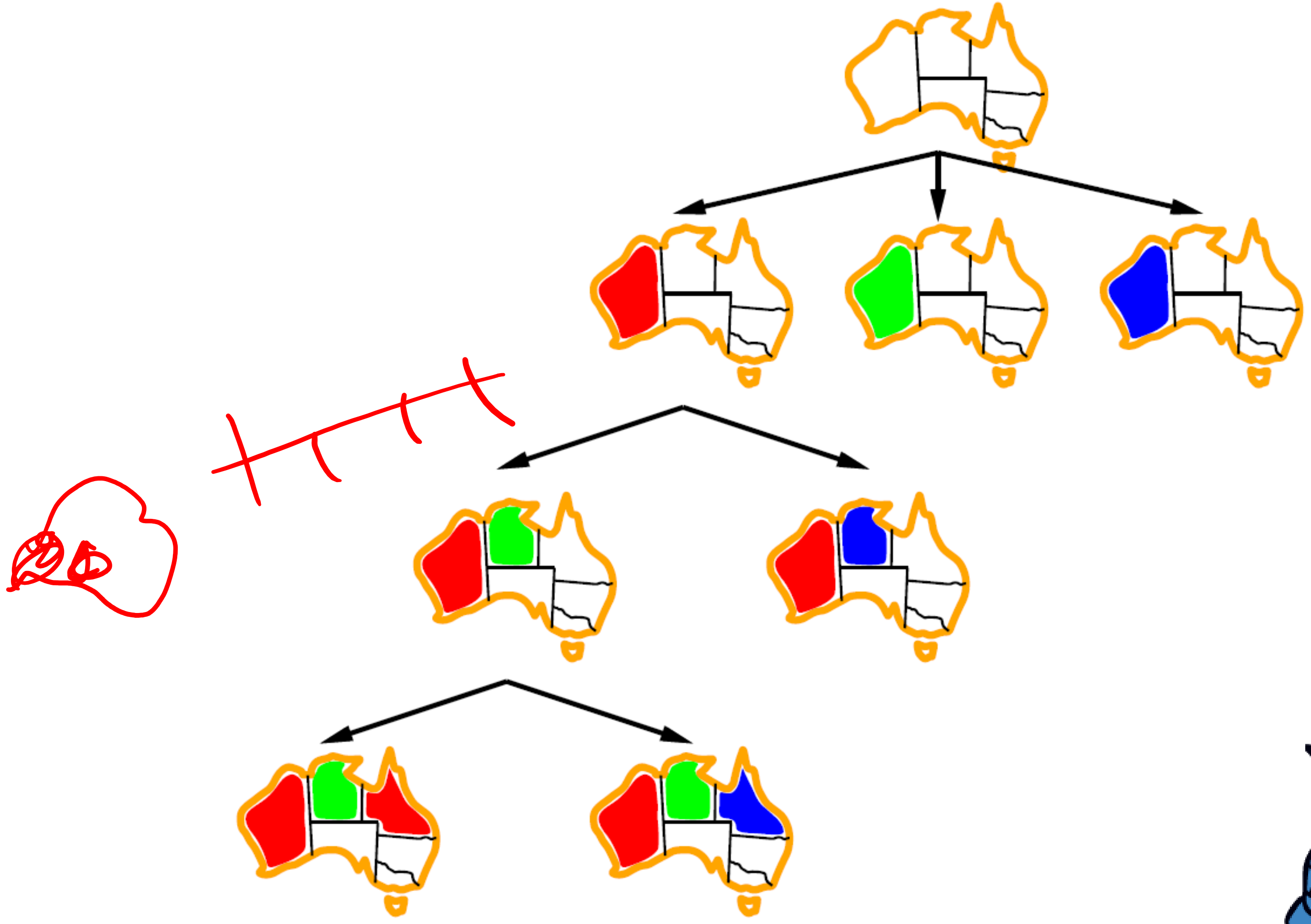
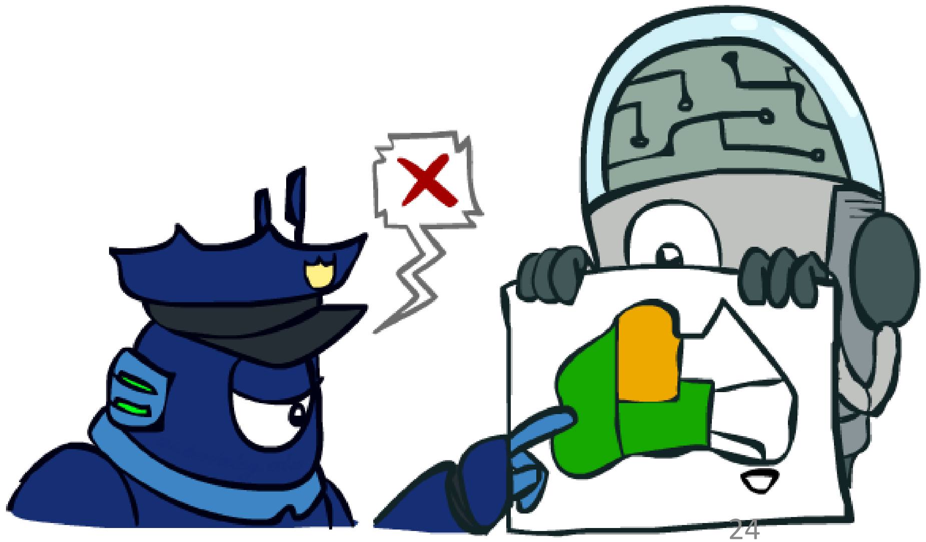
Backtracking Example





Slide/Image credit: Ketrina Yim, UCB CS 188

Backtracking Search

```
function BACKTRACKING-SEARCH(csp) returns solution/failure
    return RECURSIVE-BACKTRACKING({ }, csp)

function RECURSIVE-BACKTRACKING(assignment, csp) returns soln/failure
    if assignment is complete then return assignment
    var ← SELECT-UNASSIGNED-VARIABLE(VARIABLES[csp], assignment, csp)
    for each value in ORDER-DOMAIN-VALUES(var, assignment, csp) do
        if value is consistent with assignment given CONSTRAINTS[csp] then
            add {var = value} to assignment
            result ← RECURSIVE-BACKTRACKING(assignment, csp)
            if result ≠ failure then return result
            remove {var = value} from assignment
    return failure
```

Image credit: aima.cs.berkeley.edu

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No need to check constraints for a complete assignment

Image credit: aima.cs.berkeley.edu

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Checks consistency at each assignment

Image credit: aima.cs.berkeley.edu

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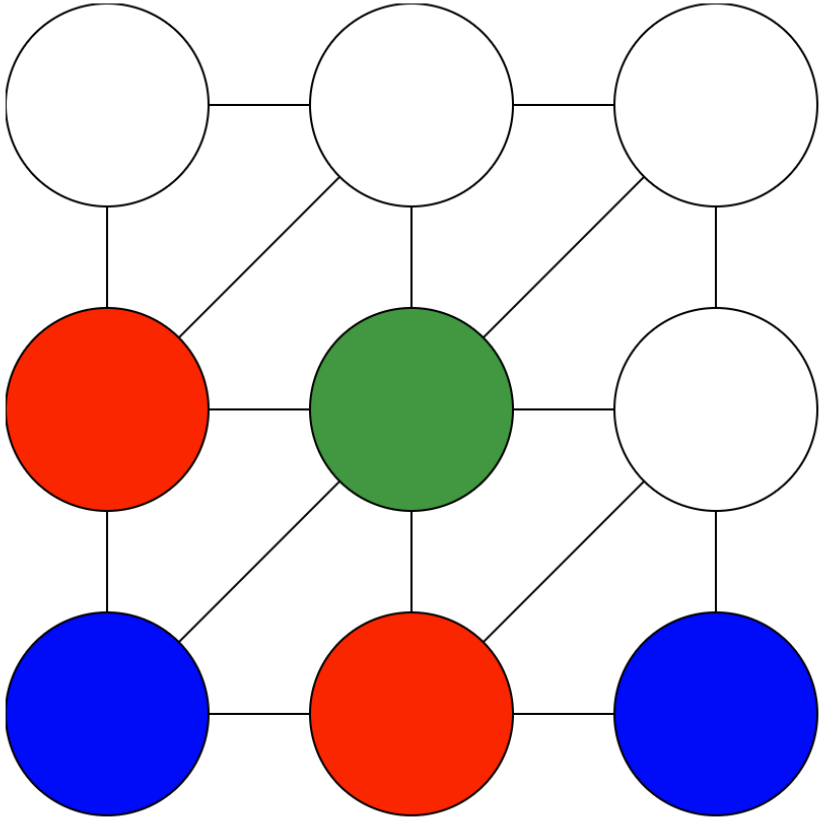
- Backtracking = DFS + variable-ordering + fail-on-violation
- What are the decision points?

Image credit: aima.cs.berkeley.edu

Demo – Backtracking

https://www.cs.cmu.edu/~15281/demos/csp_backtracking



Demo credit: UCB CS 188

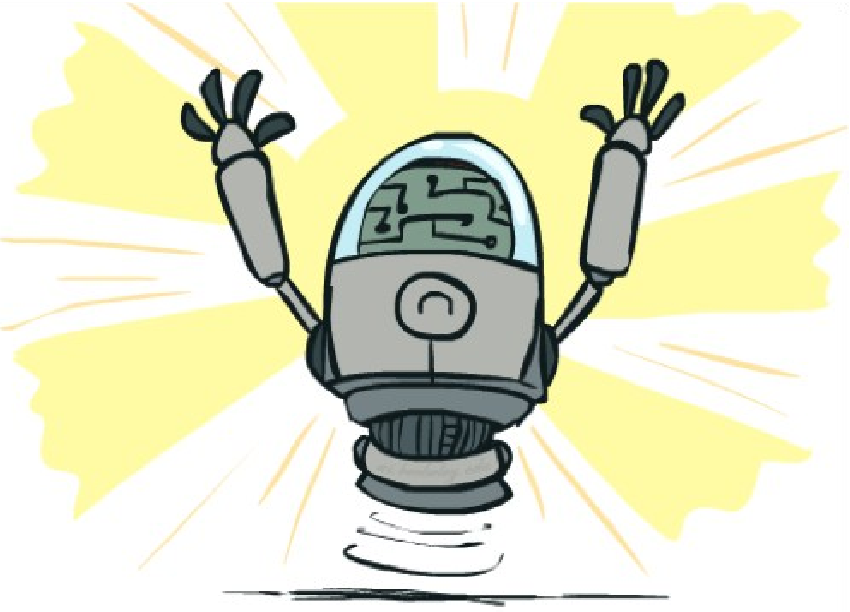
Improving Backtracking

- General-purpose ideas give huge gains in speed

- Filtering: Can we detect inevitable failure early?

- Ordering:
  - Which variable should be assigned next?
  - In what order should its values be tried?

- Structure: Can we exploit the problem structure?



Filtering

Slide/Image credit: Ketrina Yim, UCB CS 188

Filtering: Forward Checking

Filtering: Keep track of domains for unassigned variables and cross off bad options

Forward checking: A simple way for filtering

- After a variable is assigned a value, check related constraints and cross off values of unassigned variables which violate the constraints
- Failure detected if some variables have no values remaining

Filtering: Forward Checking

- Filtering: Keep track of domains for unassigned variables and cross off bad options
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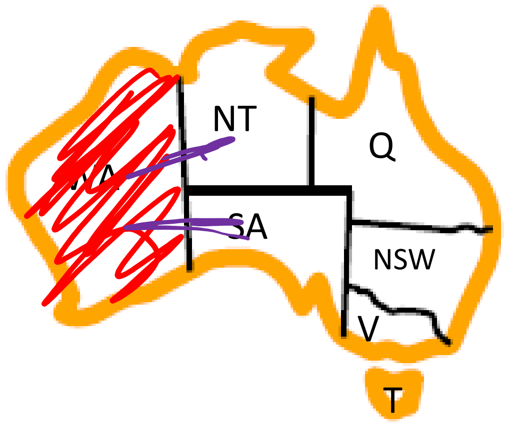
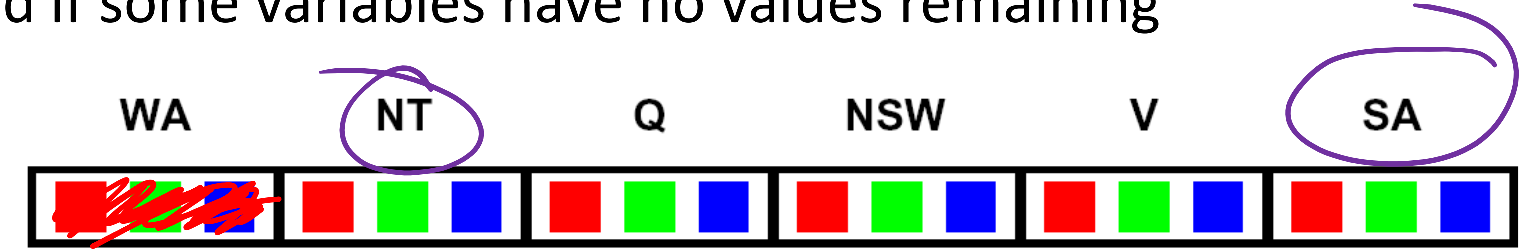
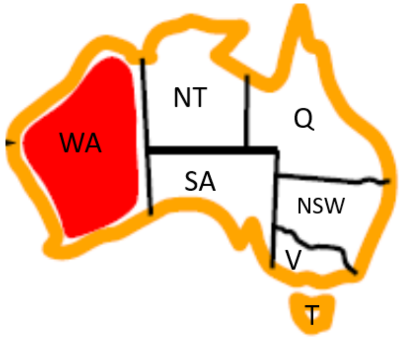
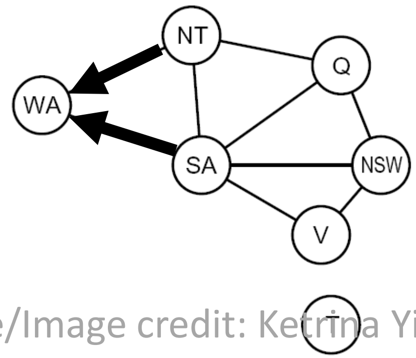




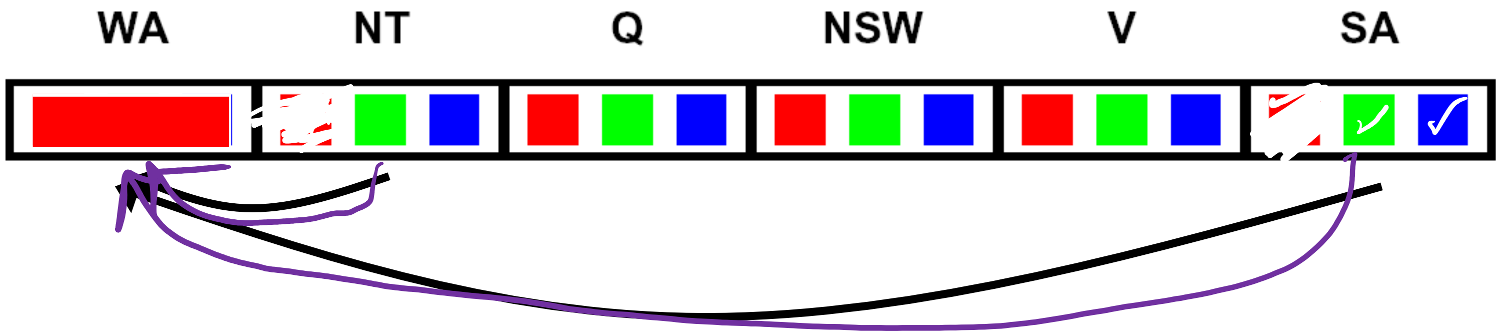
Consistency of A Single Arc

- An arc X → Y is consistent iff for every x in the tail there is some y in the head which could be assigned without violating a constraint
- Enforce arc consistency: Remove values in domain of X if no corresponding legal Y exists
- Forward checking: Only enforce $X \rightarrow Y, \forall(X, Y) \in E$ and Y newly assigned





(Remove values from the tail!)



Recall: Binary constraint graph for a binary CSP (i.e., each constraint has most two variables): nodes are variables, edges show constraints

Slide/Image credit: Ketrina Yim, UCB CS 188

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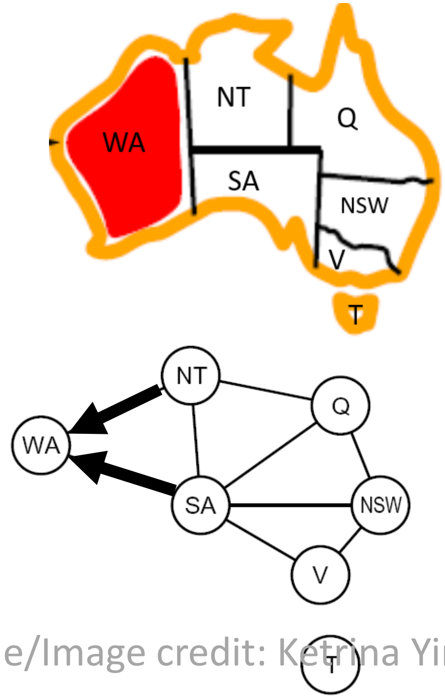

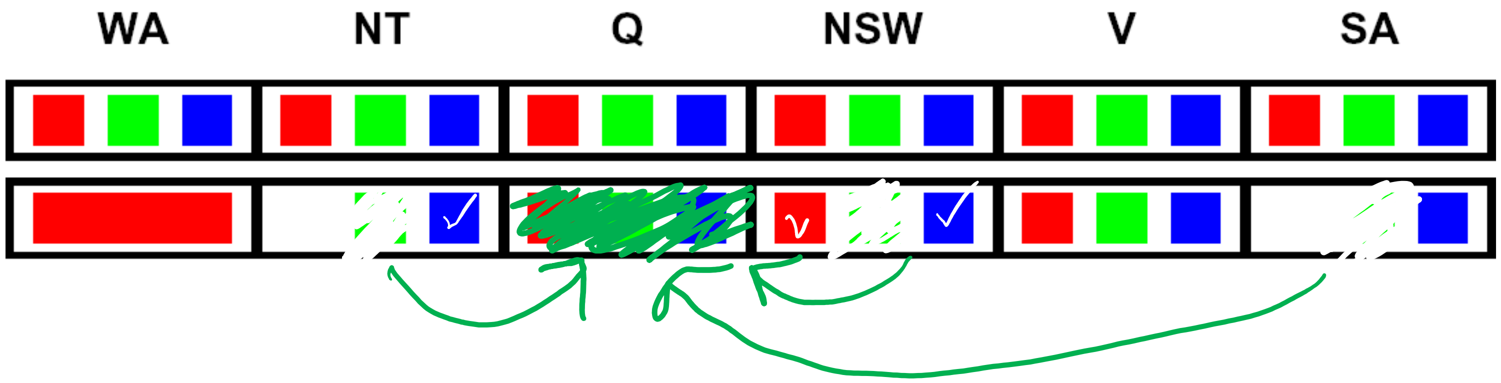

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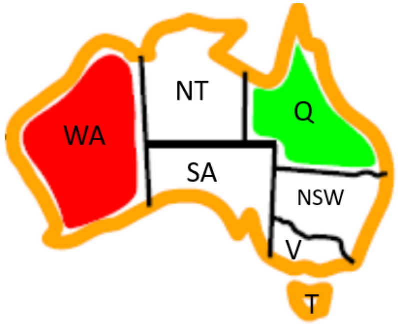
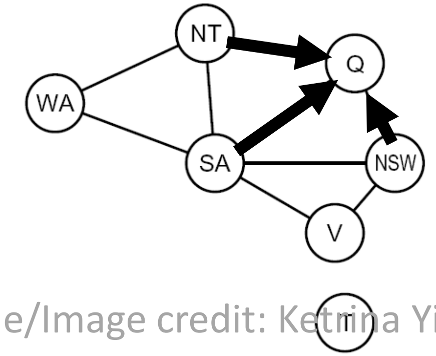
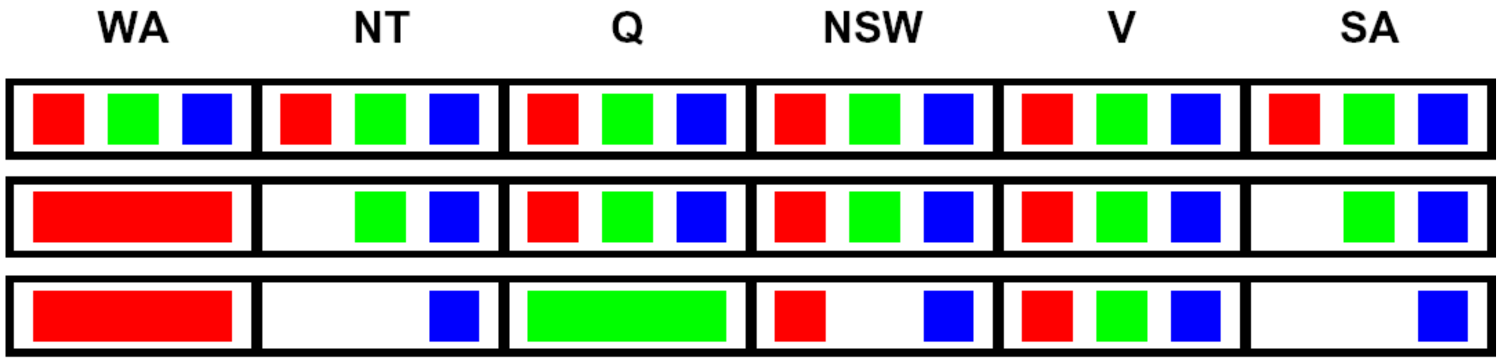
  - Failure detected if some variables have no values remaining

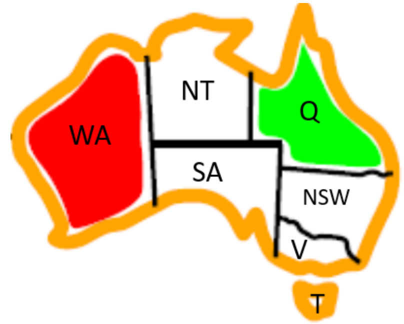


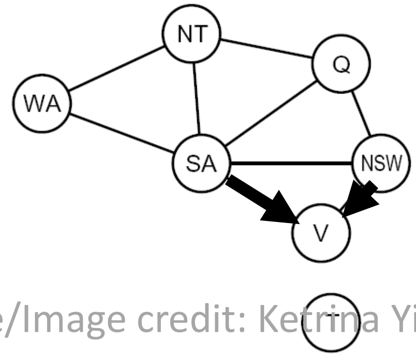


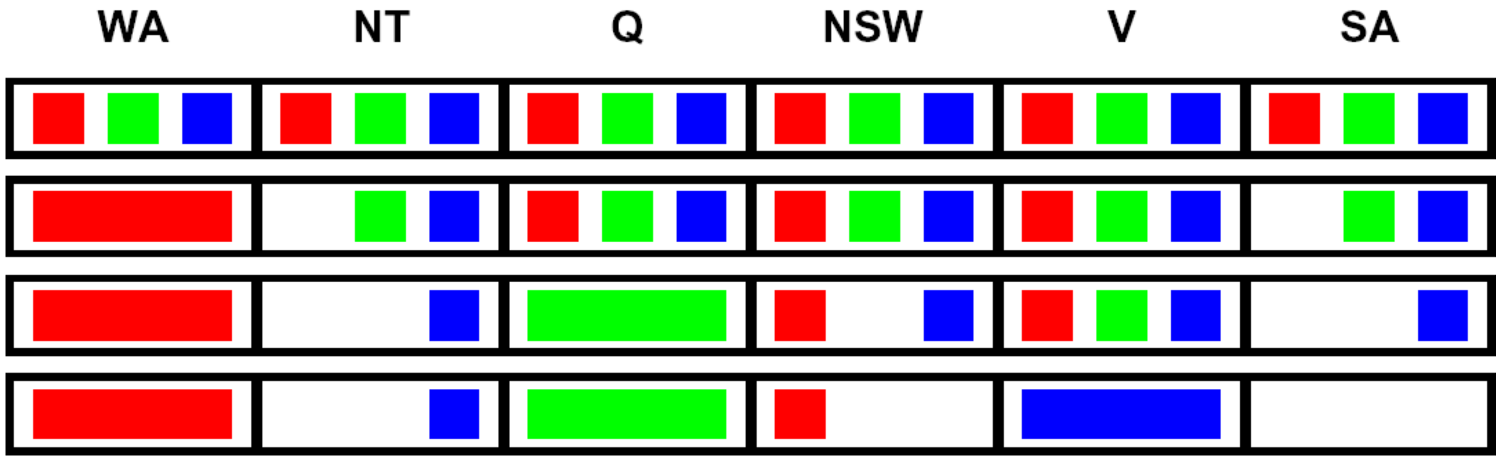


Filtering: Forward Checking

- Filtering: Keep track of domains for unassigned variables and cross off bad options
- Forward checking: A simple way for filtering
  - After a variable is assigned a value, check related constraints and cross off values of unassigned variables which violate the constraints
  - Failure detected if some variables have no values remaining





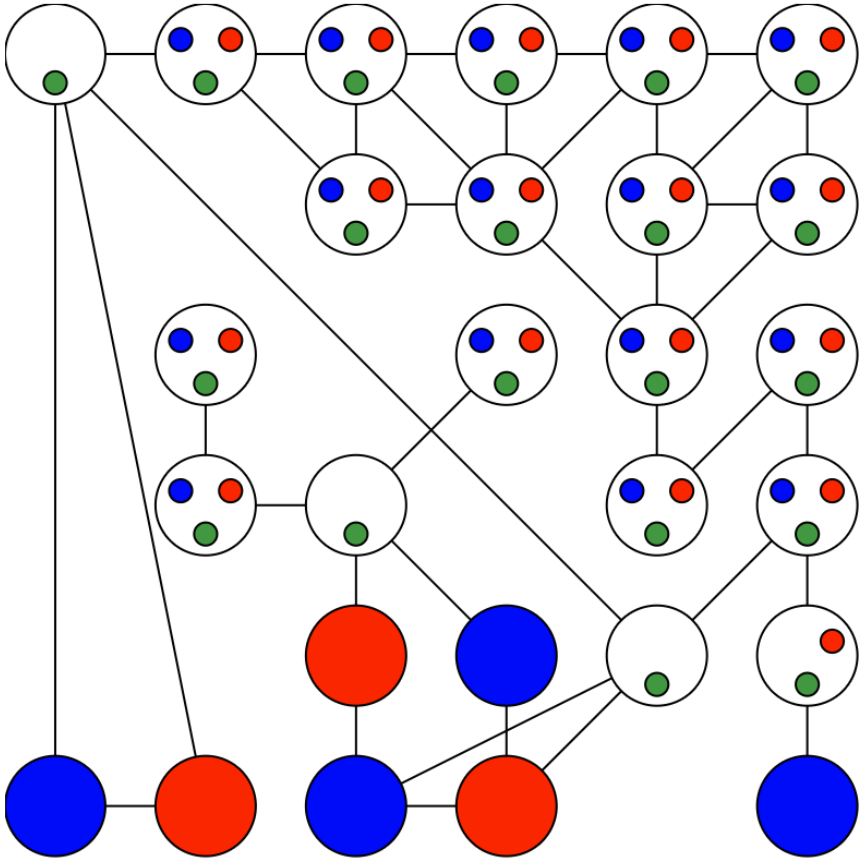


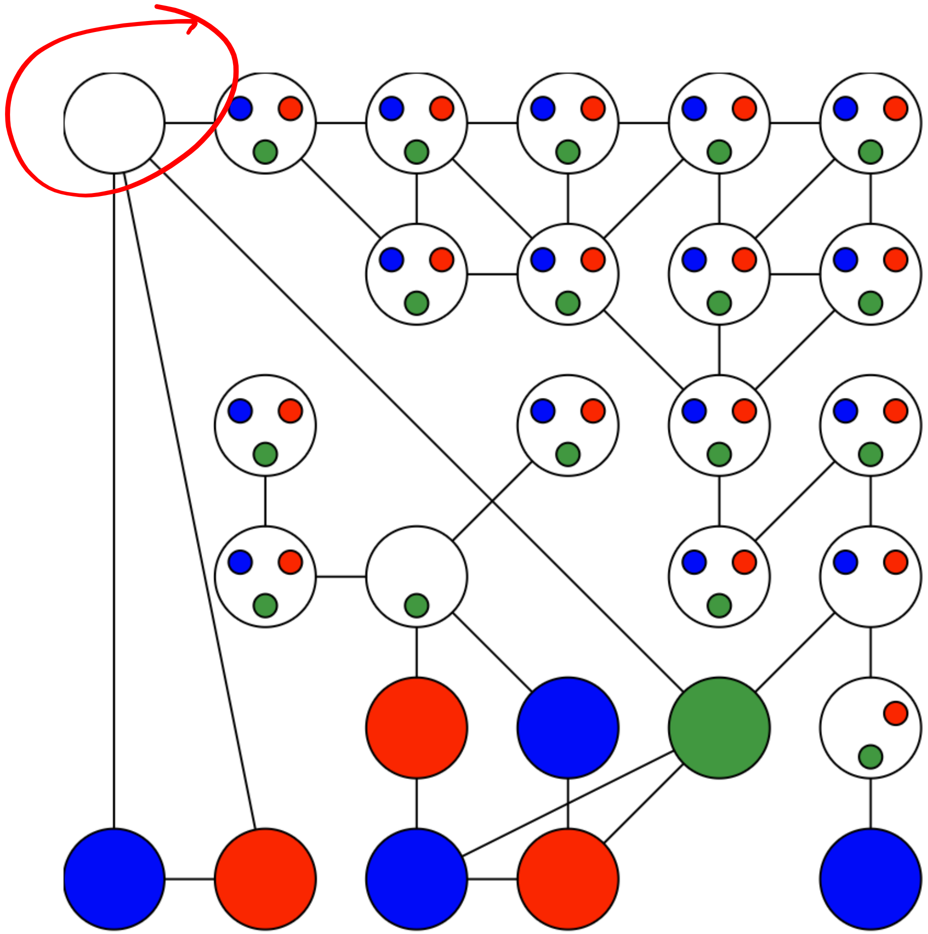
FAIL – variable with no possible values

Slide/Image credit: Ketrina Yim, UCB CS 188

Demo – Backtracking with Forward Checking

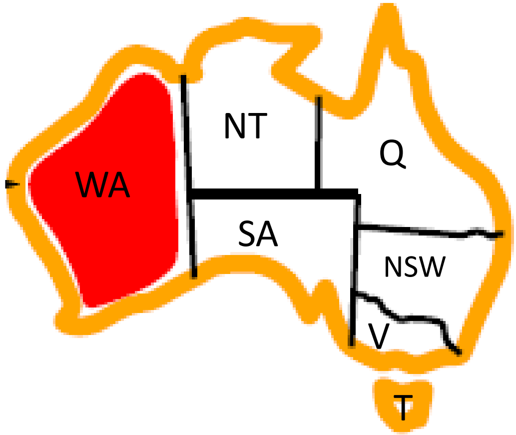
https://www.cs.cmu.edu/~15281/demos/csp_backtracking

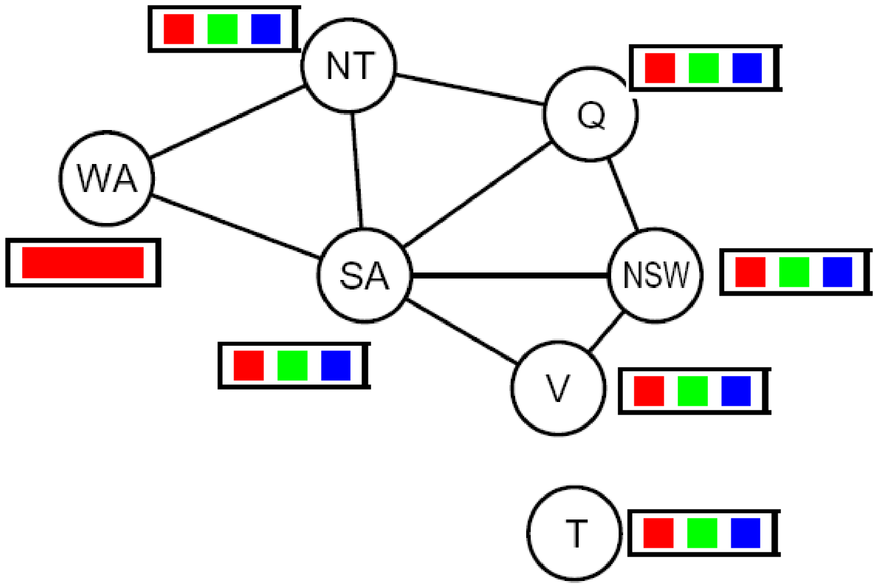




How to Enforce Arc Consistency of Entire CSP

- A simplistic algorithm: Cycle over the pairs of variables, enforcing arc-consistency, repeating the cycle until no domains change for a whole cycle
- AC-3 (short for Arc Consistency Algorithm #3): A more efficient algorithm ignoring constraints that have not been modified since they were last analyzed





AC-3: Enforce Arc Consistency of Entire CSP

```
function AC-3( csp) returns the CSP, possibly with reduced domains
   inputs: csp, a binary CSP with variables {X_1, X_2, ..., X_n}
   local variables: queue, a queue of arcs, initially all the arcs in csp

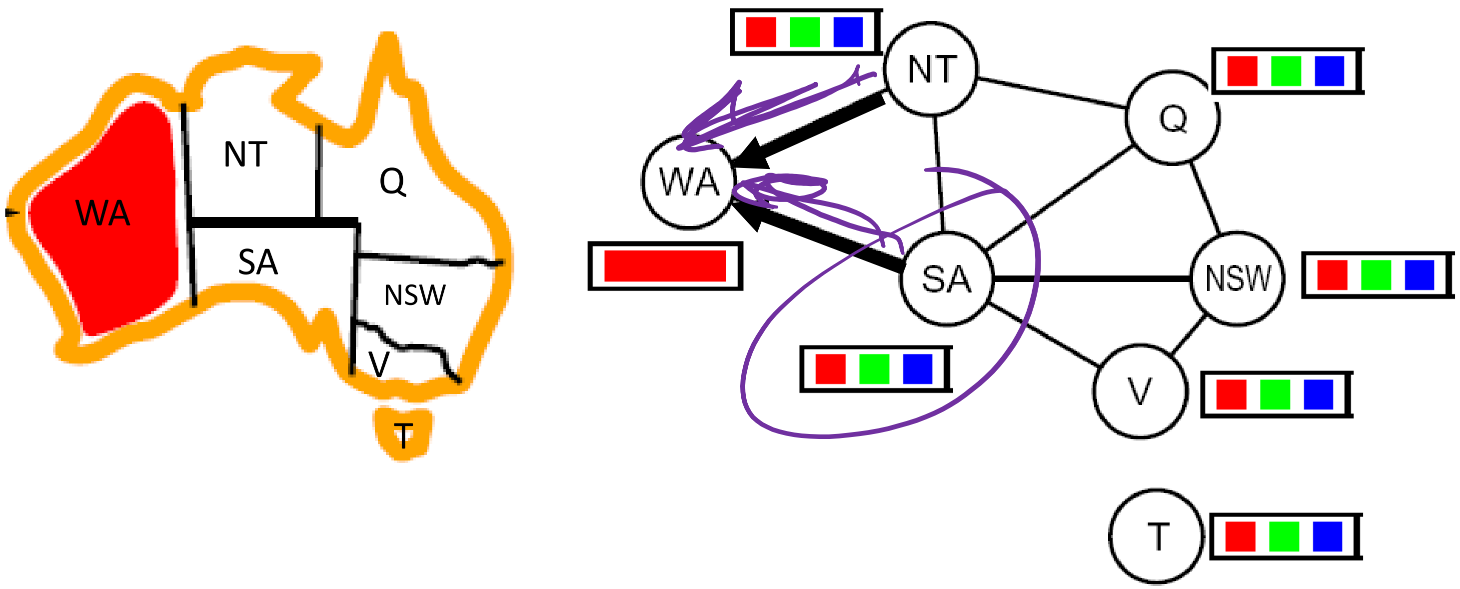
   while queue is not empty do
      (X_i, X_j) ← REMOVE-FIRST(queue)
      if REMOVE-INCONSISTENT-VALUES(X_i, X_j) then
         for each X_k in NEIGHBORS[X_i] do
            add (X_k, X_i) to queue
```

```
function REMOVE-INCONSISTENT-VALUES( X_i, X_j) returns true iff succeeds
   removed ← false
   for each x in DOMAIN[X_i] do
      if no value y in DOMAIN[X_j] allows (x,y) to satisfy the constraint X_i ↔ X_j
         then delete x from DOMAIN[X_i]; removed ← true
   return removed
```

Constraint Propagation!

Image credit: aima.cs.berkeley.edu

AC-3: Enforce Arc Consistency of Entire CSP

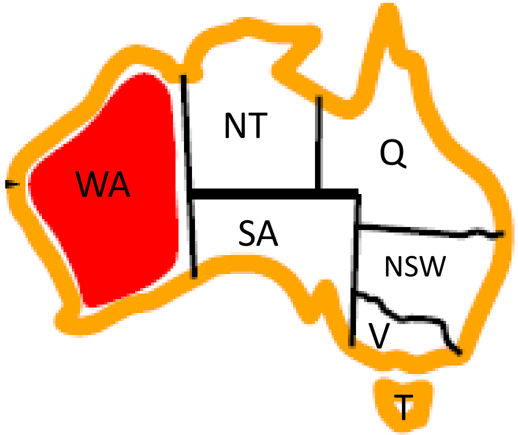


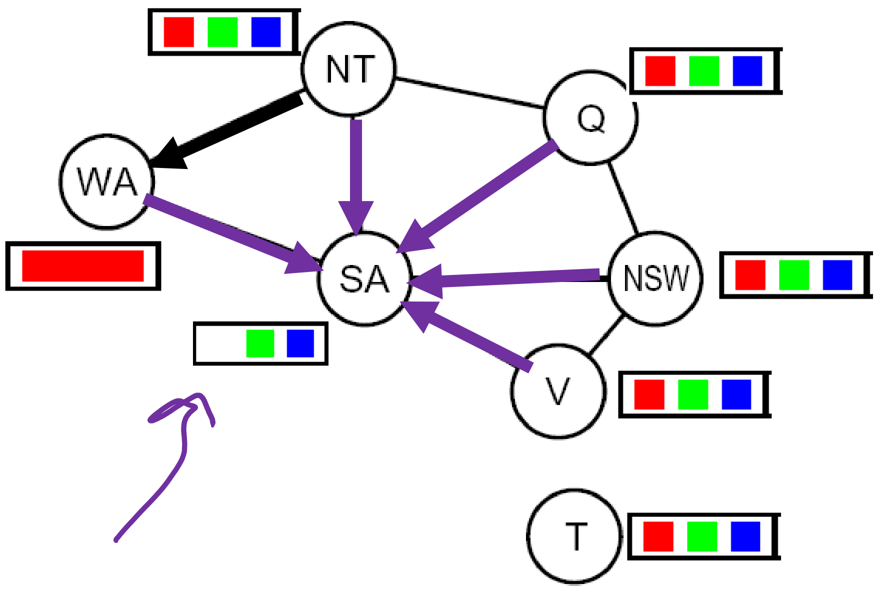
Queue:
SA->WA
NT->WA

Remember: Delete from the tail!

Image credit: Ketrina Yim, UCB CS 188

AC-3: Enforce Arc Consistency of Entire CSP



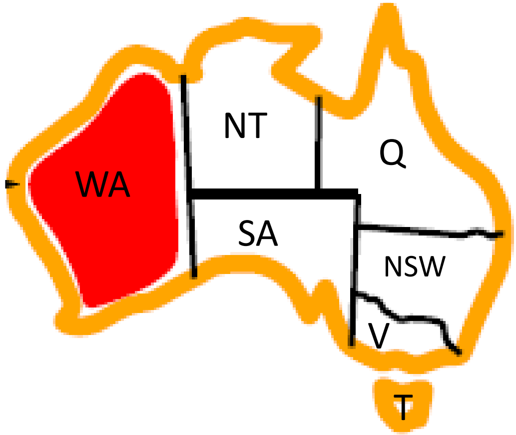


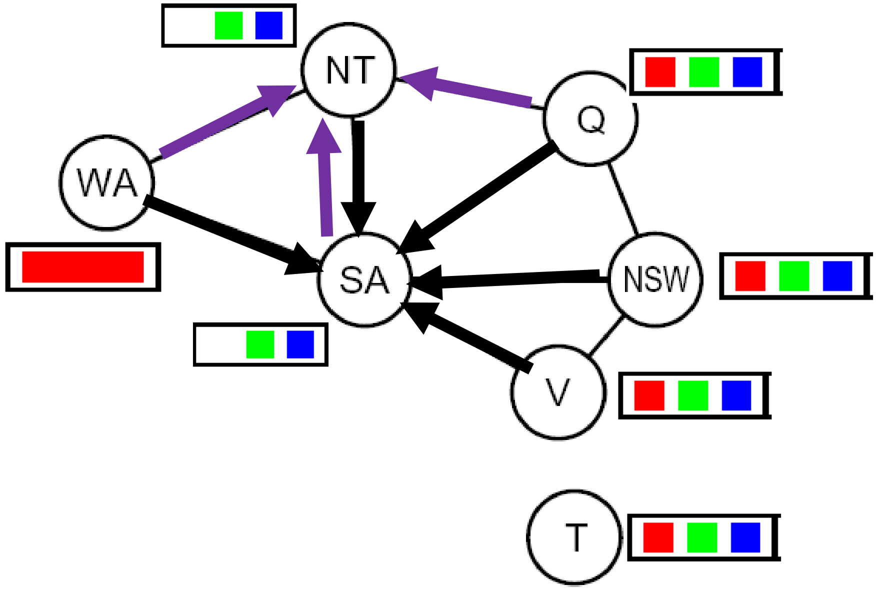
Queue:
NT->WA
WA->SA
NT->SA
Q->SA
NSW->SA
V->SA

Image credit: Ketrina Yim, UCB CS 188

Remember: *Delete from the tail!*

AC-3: Enforce Arc Consistency of Entire CSP



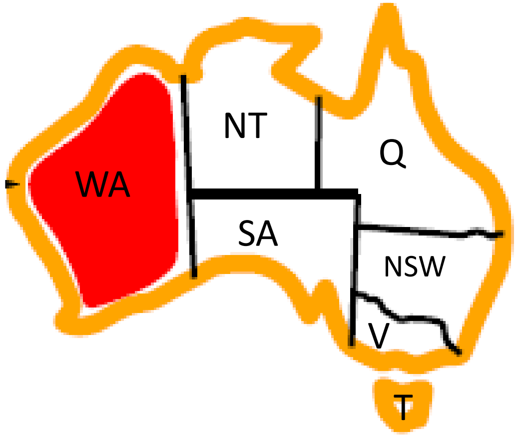


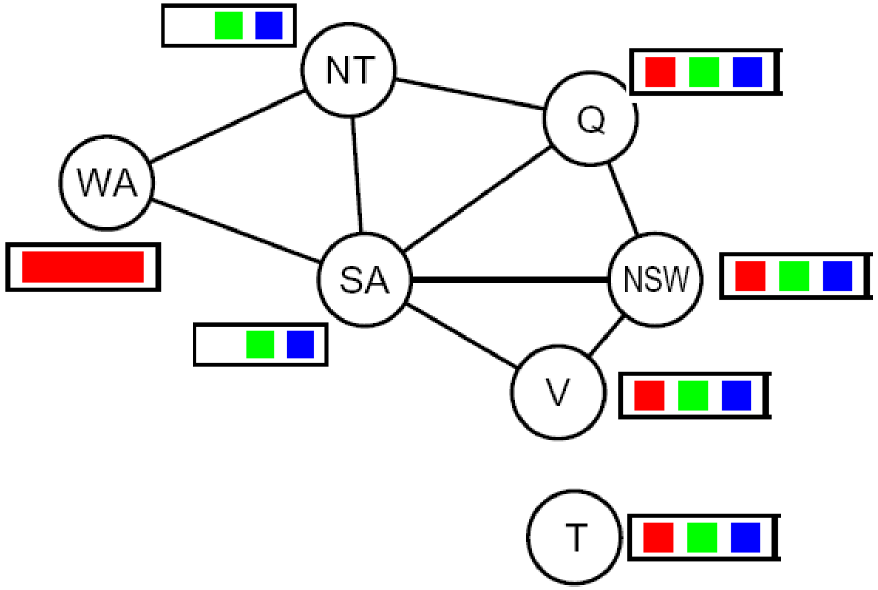
Queue:
WA->SA
NT->SA
Q->SA
NSW->SA
V->SA
WA->NT
SA->NT
Q->NT

Remember: Delete from the tail!

Image credit: Ketrina Yim, UCB CS 188

AC-3: Enforce Arc Consistency of Entire CSP



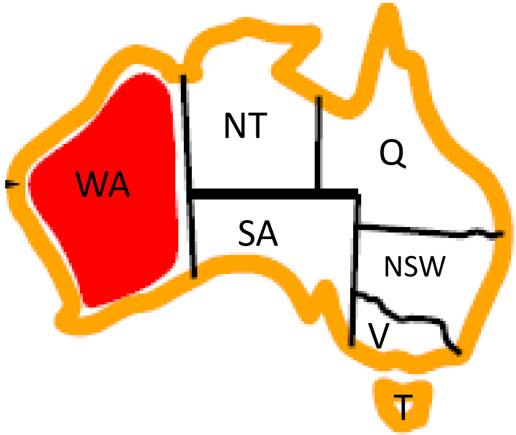


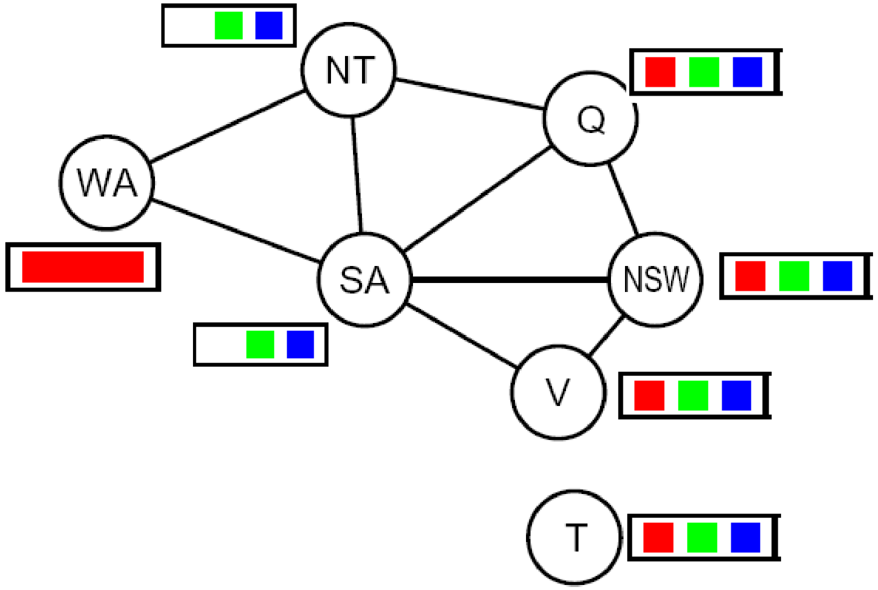
Queue:
WA->SA
NT->SA
Q->SA
NSW->SA
V->SA
WA->NT
SA->NT
Q->NT

Remember: Delete from the tail!

Image credit: Ketrina Yim, UCB CS 188

AC-3: Enforce Arc Consistency of Entire CSP





Queue:

NT->SA

Q->SA

NSW->SA

V->SA

WA->NT

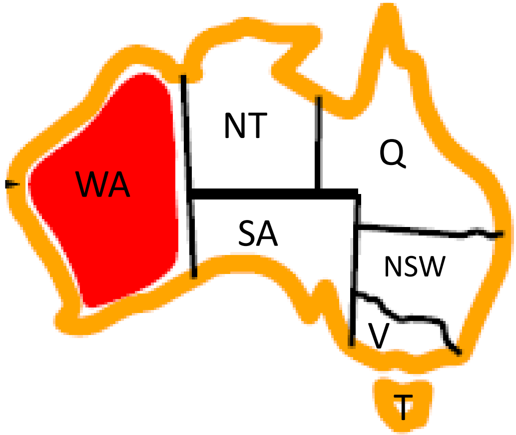
SA->NT

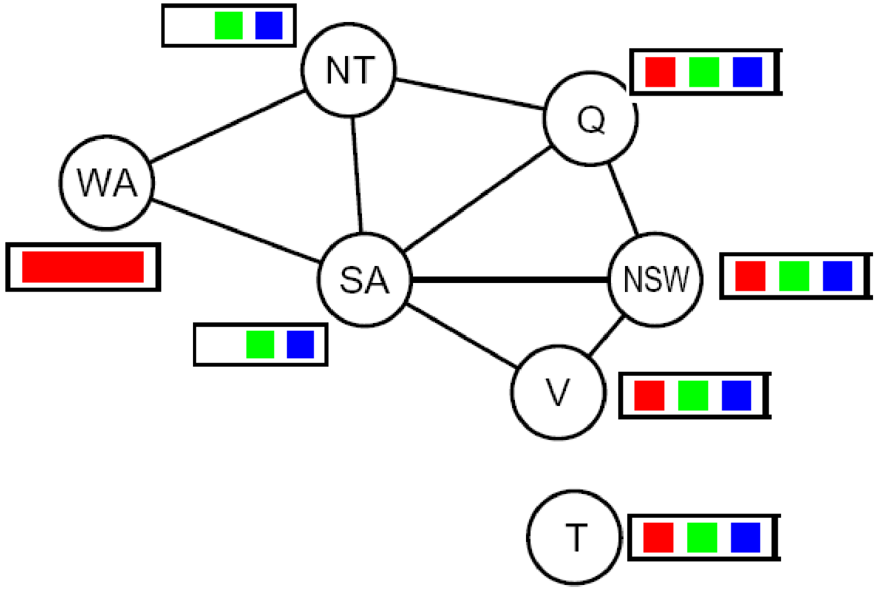
Q->NT

Image credit: Ketrina Yim, UCB CS 188

Remember: Delete from the tail!

AC-3: Enforce Arc Consistency of Entire CSP



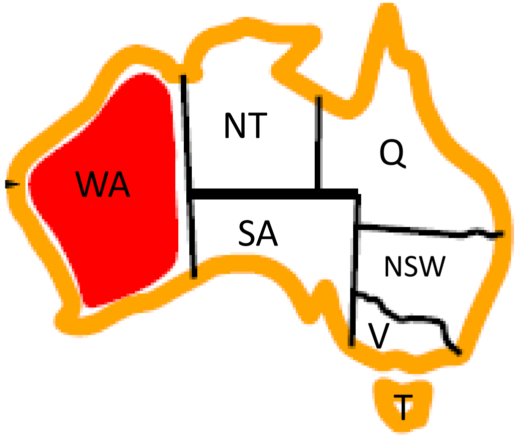


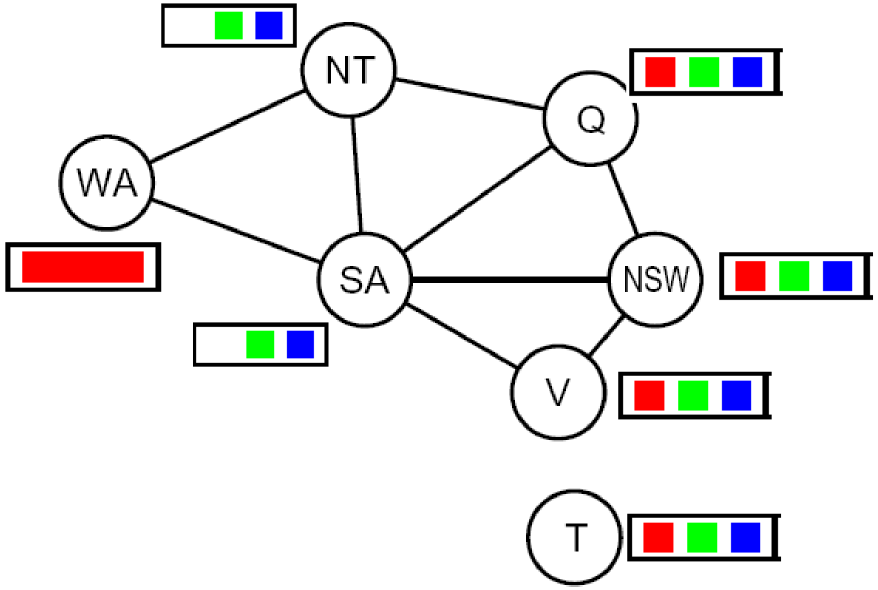
Queue:
Q->SA
NSW->SA
V->SA
WA->NT
SA->NT
Q->NT

Image credit: Ketrina Yim, UCB CS 188

Remember: Delete from the tail!

AC-3: Enforce Arc Consistency of Entire CSP





Queue:

NSW->SA

V->SA

WA->NT

SA->NT

Q->NT

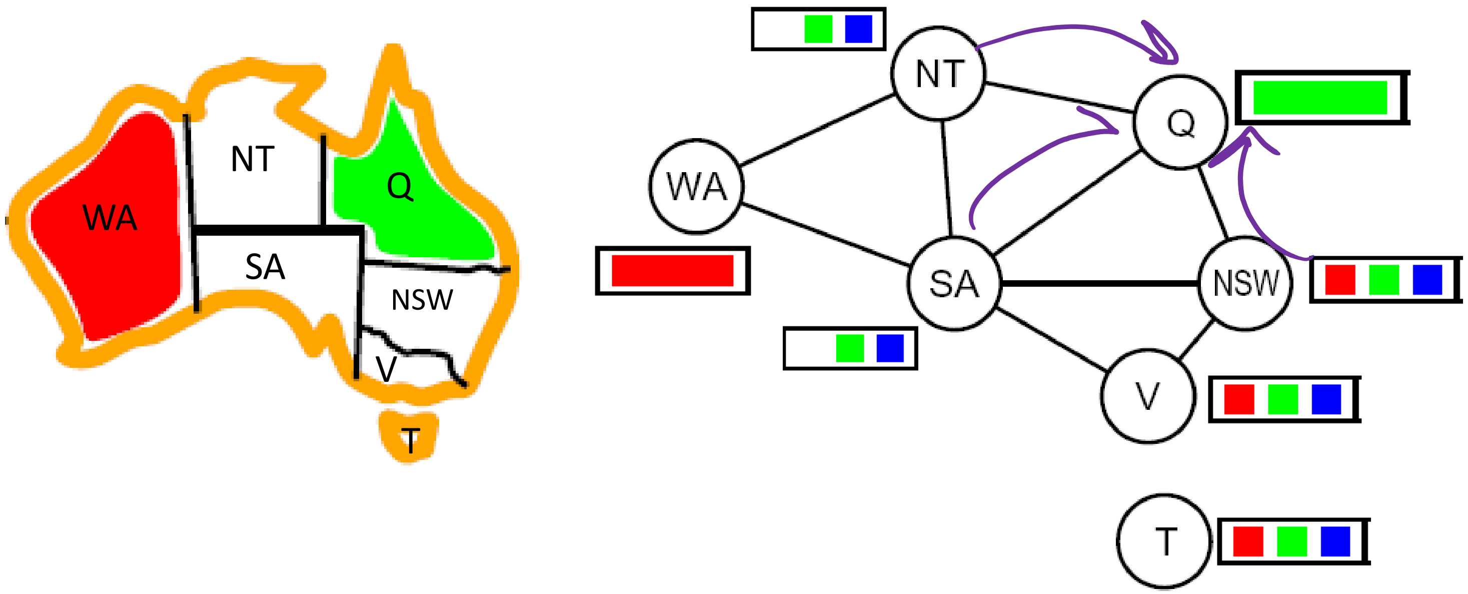
All clean

Remember: Delete from the tail!

Image credit: Ketrina Yim, UCB CS 188

AC-3: Enforce Arc Consistency of Entire CSP

Now pick
Q = green

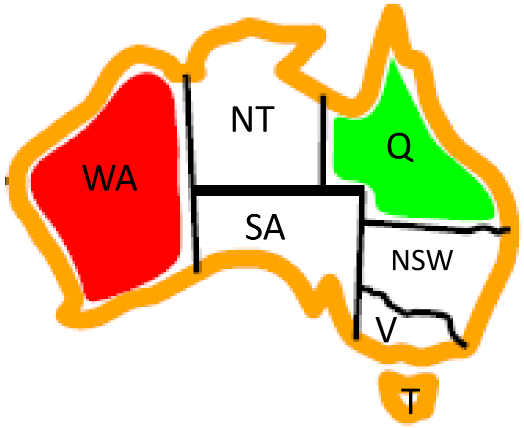


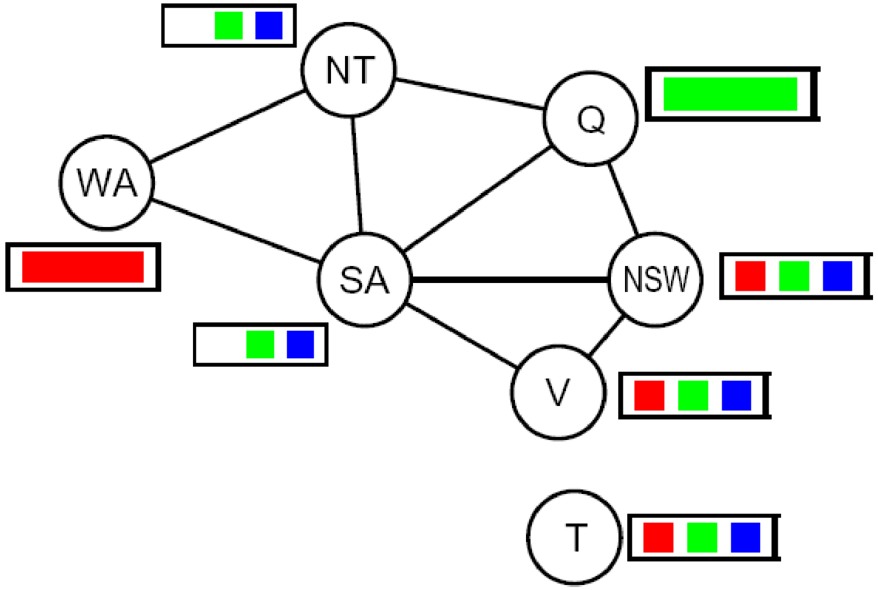
Queue:

Remember: Delete from the tail!

Image credit: Ketrina Yim, UCB CS 188

Poll 2: After assigning Q to Green, what gets added to the Queue?





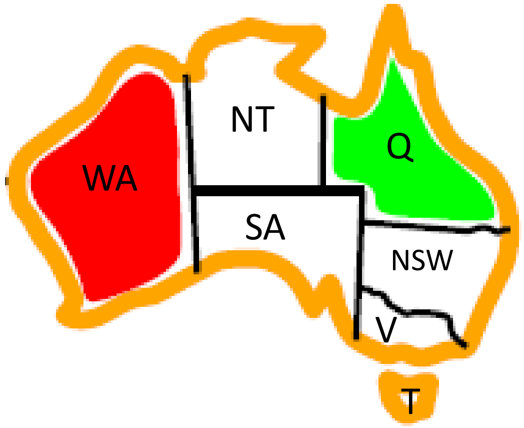
Queue:

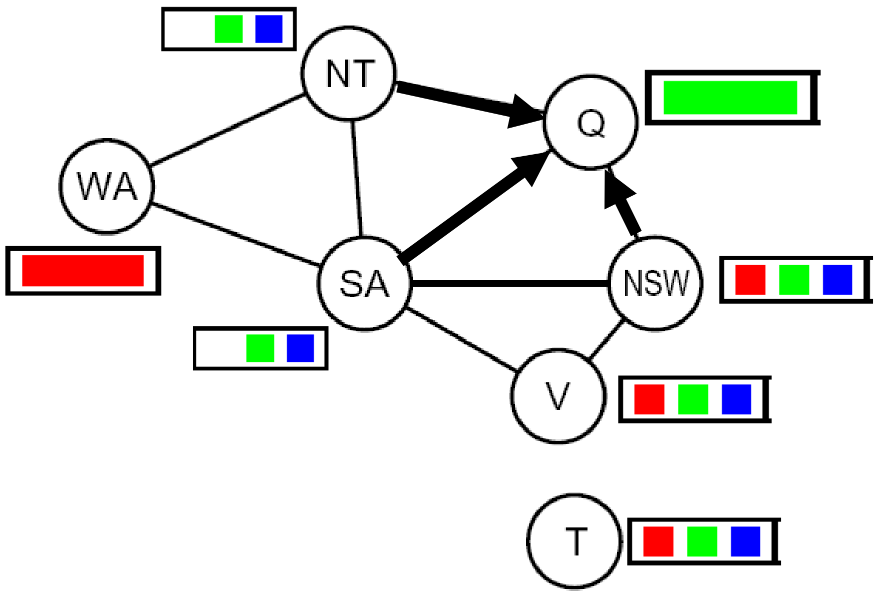
A: NSW->Q, SA->Q, NT->Q

B: Q->NSW, Q->SA, Q->NT

Image credit: Ketrina Yim, UCB CS 188

AC-3: Enforce Arc Consistency of Entire CSP



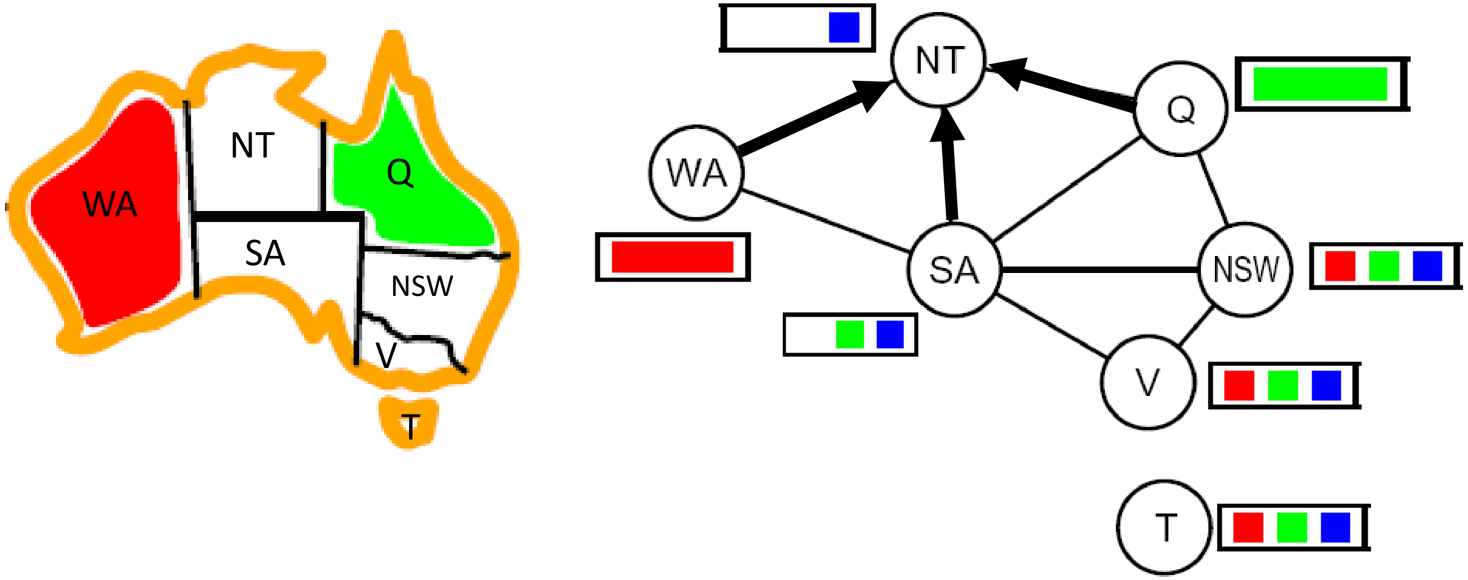


Queue:
NT->Q
SA->Q
NSW->Q

Image credit: Ketrina Yim, UCB CS 188

Remember: Delete from the tail!

AC-3: Enforce Arc Consistency of Entire CSP

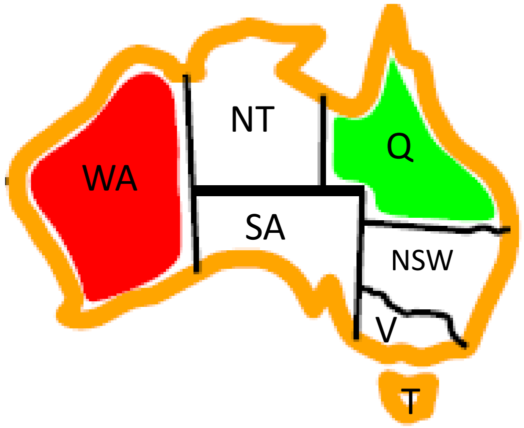


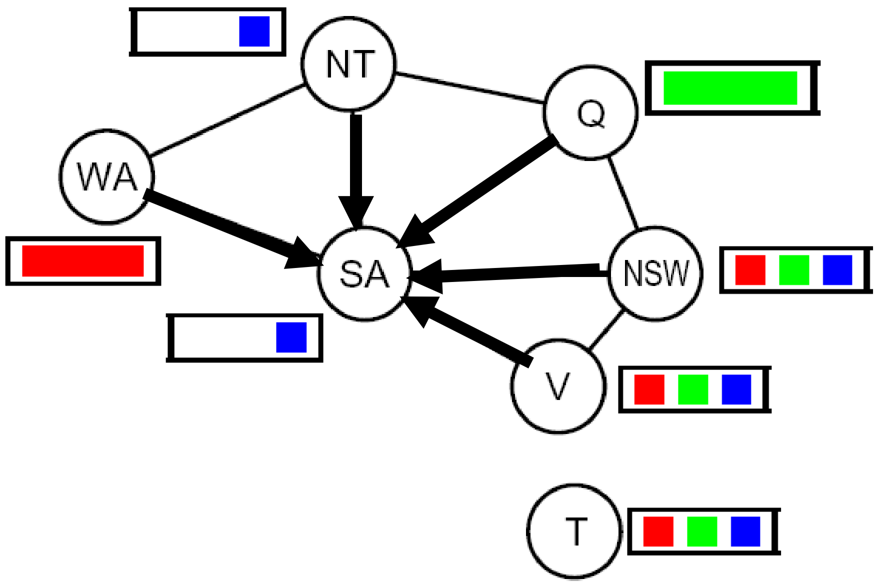
Queue:
SA->Q
NSW->Q
WA->NT
SA->NT
Q->NT

Remember: Delete from the tail!

Image credit: Ketrina Yim, UCB CS 188

AC-3: Enforce Arc Consistency of Entire CSP



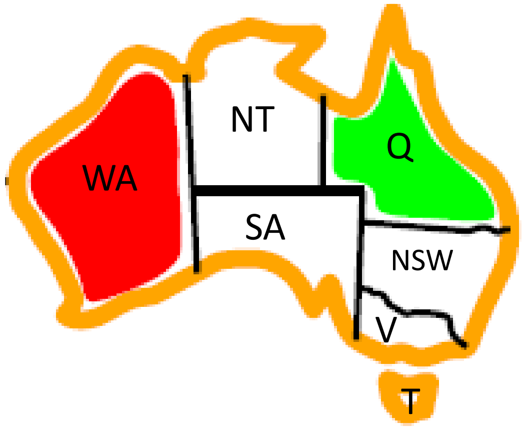


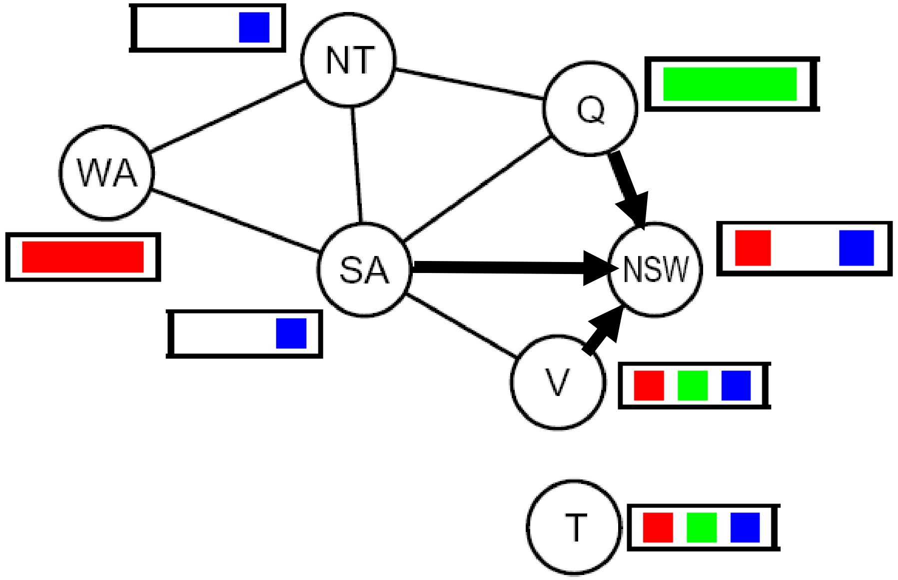
Queue:
NSW->Q
WA->NT
SA->NT
Q->NT
WA->SA
NT->SA
Q->SA
NSW->SA
V->SA

Image credit: Ketrina Yim, UCB CS 188

Remember: Delete from the tail!

AC-3: Enforce Arc Consistency of Entire CSP





Queue:

WA->NT

SA->NT

Q->NT

WA->SA

NT->SA

Q->SA

NSW->SA

V->SA

V->NSW

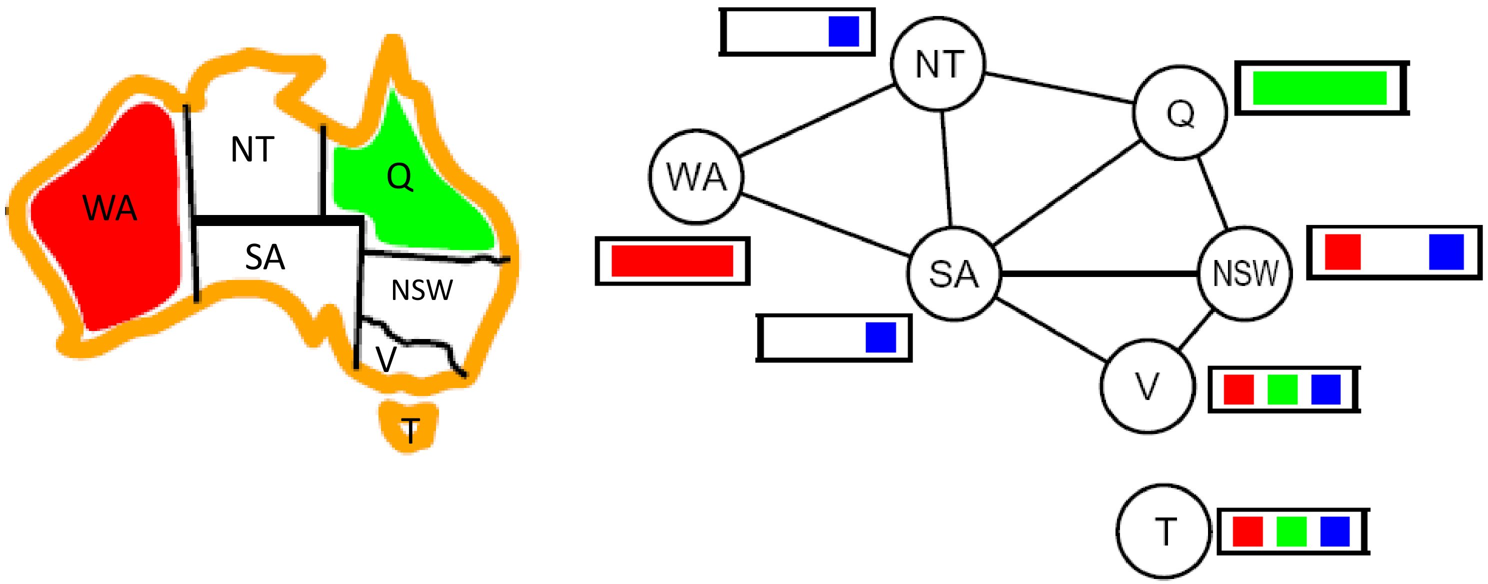
Q->NSW

SA->NSW

Remember: Delete from the tail!

Image credit: Ketrina Yim, UCB CS 188

AC-3: Enforce Arc Consistency of Entire CSP



Queue:

WA->NT

SA->NT

Q->NT

WA->SA

NT->SA

Q->SA

NSW->SA

V->SA

V->NSW

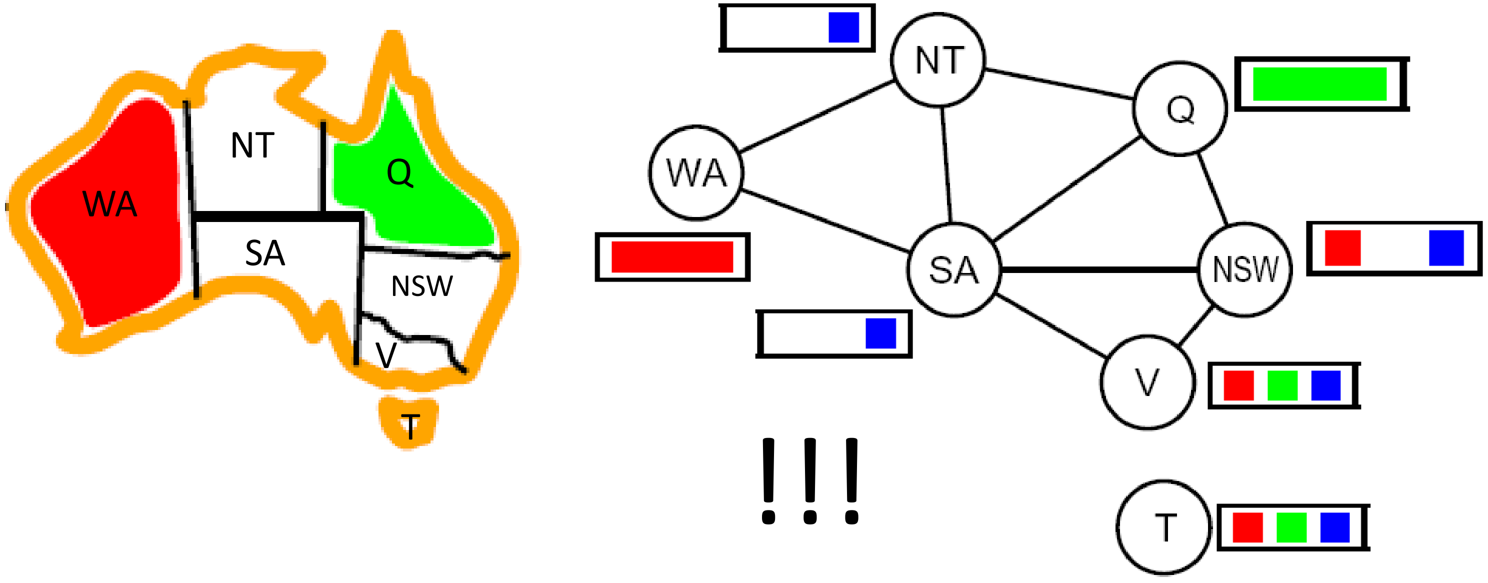
Q->NSW

SA->NSW

Remember: Delete from the tail!

Image credit: Ketrina Yim, UCB CS 188

AC-3: Enforce Arc Consistency of Entire CSP



Queue:

SA->NT

Q->NT

WA->SA

NT->SA

Q->SA

NSW->SA

V->SA

V->NSW

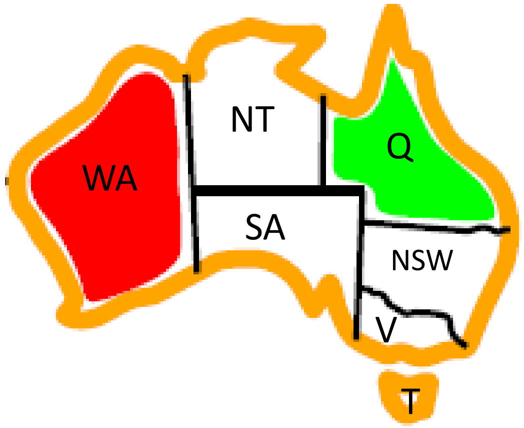
Q->NSW

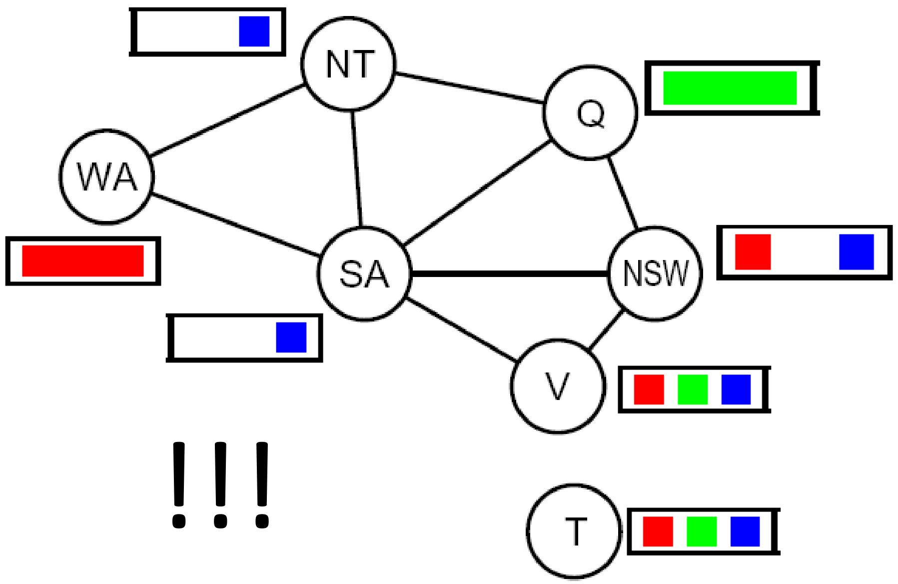
SA->NSW

Image credit: Ketrina Yim, UCB CS 188

Remember: Delete from the tail!

AC-3: Enforce Arc Consistency of Entire CSP





Queue:
SA->NT
Q->NT
WA->SA
NT->SA
Q->SA
NSW->SA
V->SA
V->NSW
Q->NSW
SA->NSW

- Backtrack on the assignment of Q
- Arc consistency detects failure earlier than forward checking
- Can be run as a preprocessor or after each assignment
- What's the downside of enforcing arc consistency?

Remember: Delete from the tail!

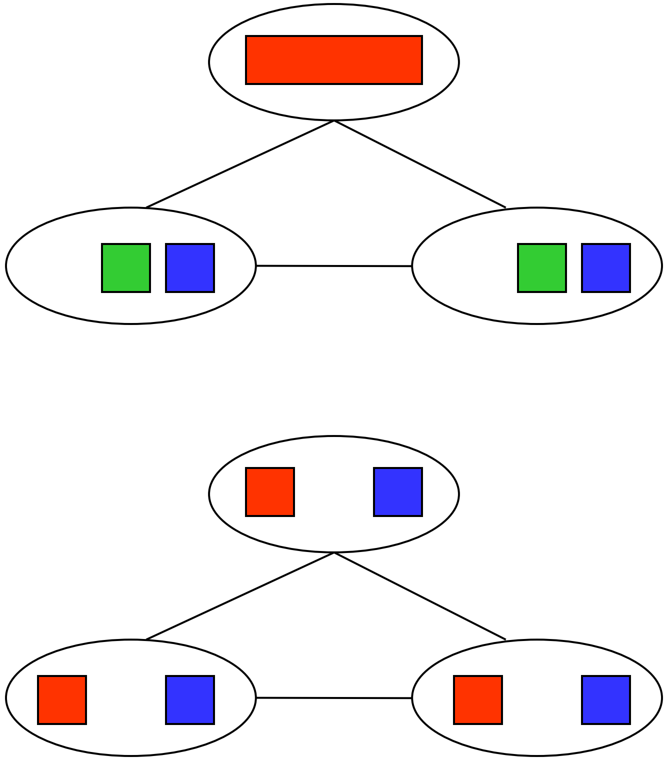
Image credit: Ketrina Yim, UCB CS 188

Limitations of Arc Consistency

- After enforcing arc consistency:
  - Can have one solution left
  - Can have multiple solutions left
  - Can have no solutions left (and not know it)

- Arc consistency only checks local consistency conditions

- Arc consistency still runs inside a backtracking search!



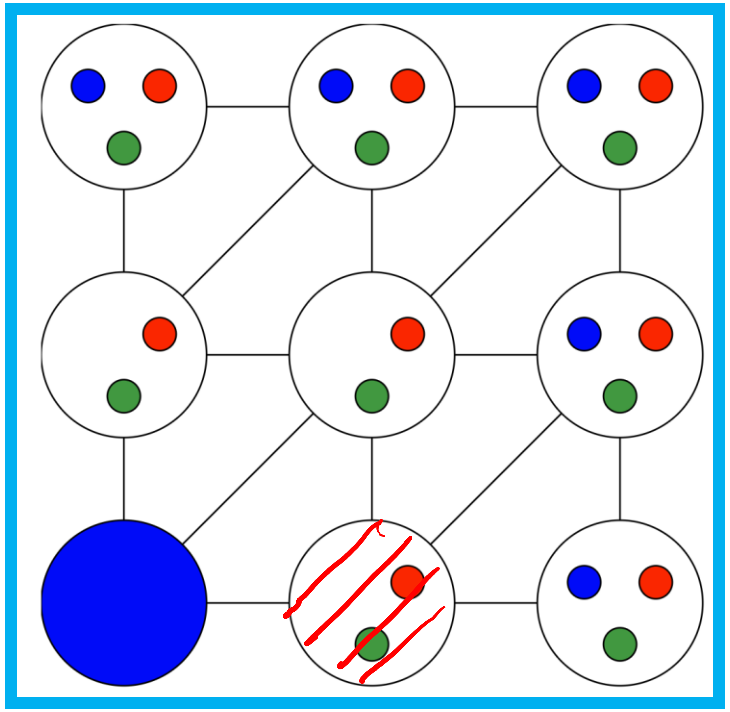
What went wrong here?

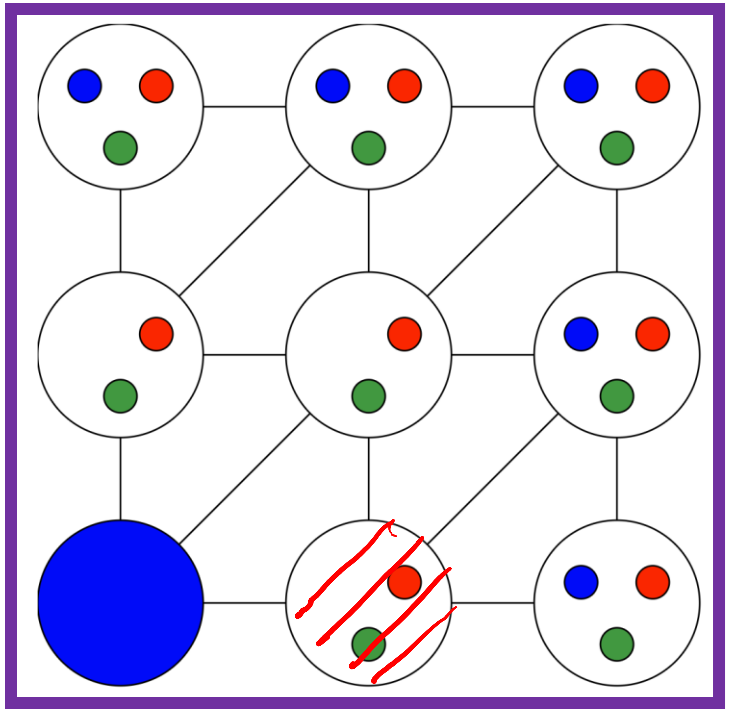
Demo – Backtracking with AC-3

Compare
- Backtracking with Forward Checking
- Backtracking with AC-3

Forward checking only check arcs connecting variables a variable that we just assigned.

With AC-3, we make sure all arcs are consistent





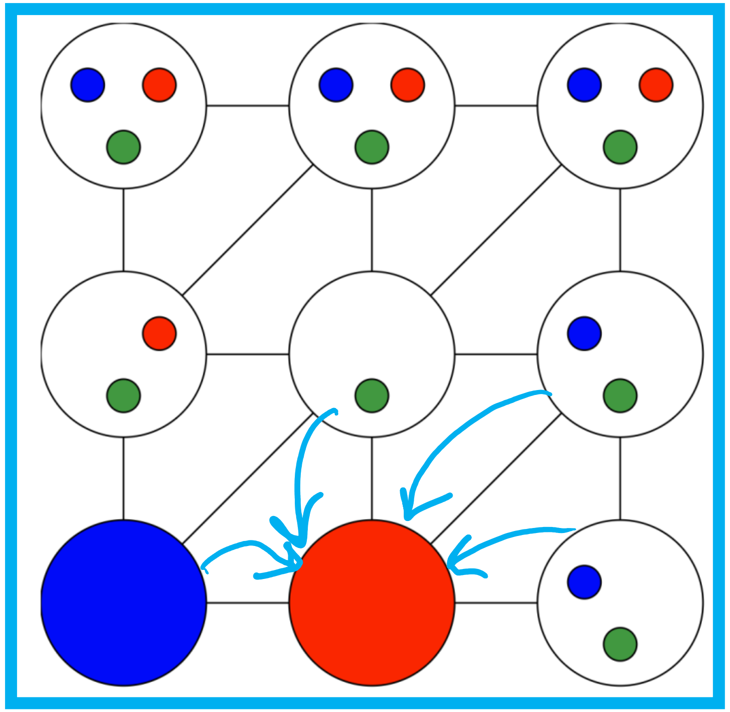
Demo credit: UCB CS 188

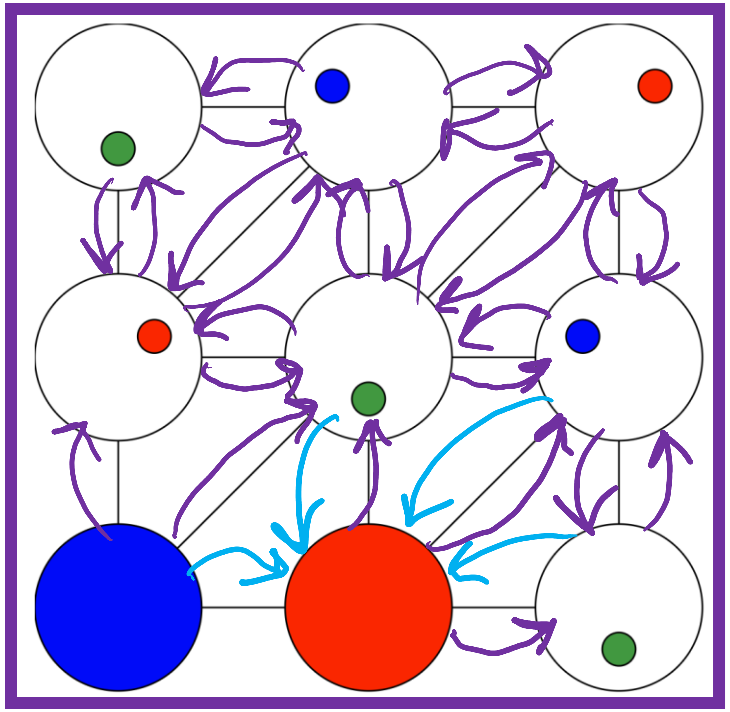
Demo – Backtracking with AC-3

Compare
- Backtracking with Forward Checking
- Backtracking with AC-3

Forward checking only check arcs connecting variables a variable that we just assigned.

With AC-3, we make sure all arcs are consistent





Demo credit: UCB CS 188

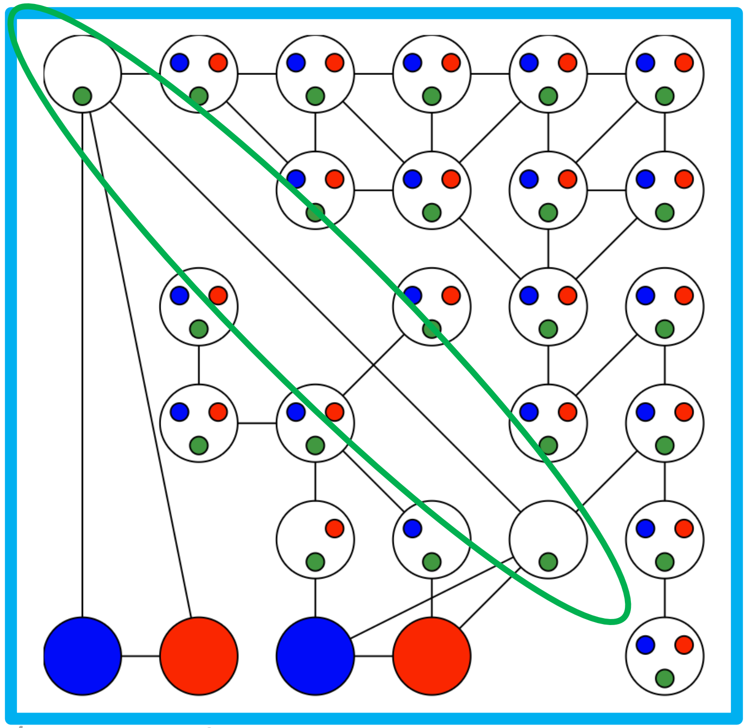
Demo – Backtracking with AC-3

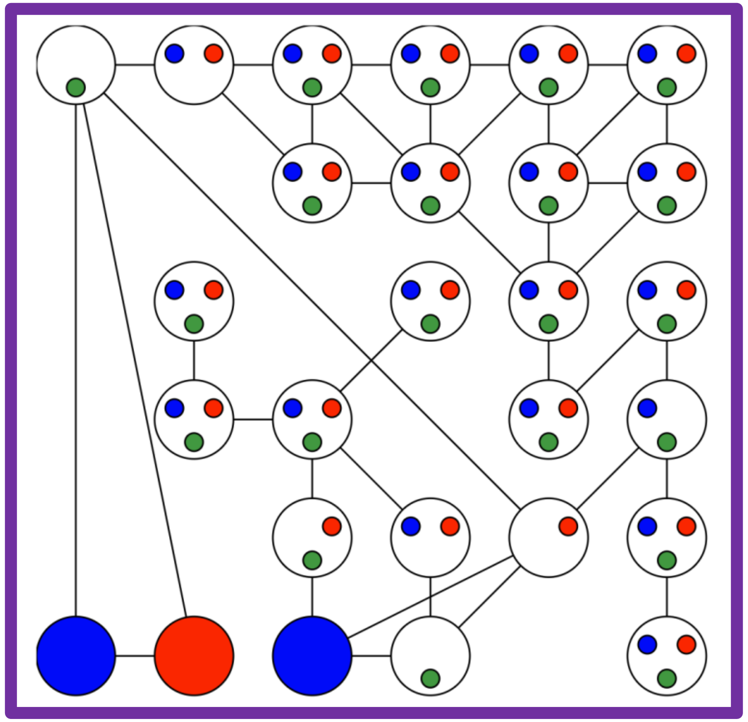
Compare

- Backtracking with Forward Checking
- Backtracking with AC-3

Forward checking only check arcs connecting variables a variable that we just assigned.

With AC-3, we can find existing problems, such as the arc between these two variables with only green left.





Complexity of a single run of AC-3

```
function AC-3( csp) returns the CSP, possibly with reduced domains
   inputs: csp, a binary CSP with variables {X1, X2, ..., Xn}
   local variables: queue, a queue of arcs, initially all the arcs in csp

   while queue is not empty do
      (Xi, Xj) ← REMOVE-FIRST(queue)
      if REMOVE-INCONSISTENT-VALUES(Xi, Xj) then
         for each Xk in NEIGHBORS[Xi] do
            add (Xk, Xi) to queue
```

```
function REMOVE-INCONSISTENT-VALUES( Xi, Xj) returns true iff succeeds
   removed ← false
   for each x in DOMAIN[Xi] do
      if no value y in DOMAIN[Xj] allows (x,y) to satisfy the constraint Xi ↔ Xj
         then delete x from DOMAIN[Xi];  removed ← true
   return removed
```

Recall that the whole backtracking algorithm with AC-3 will call AC-3 many times

Image credit: aima.cs.berkeley.edu

Complexity of a single run of AC-3

function AC-3(*csp*) **returns** the CSP, possibly with reduced domains
  inputs: *csp*, a binary CSP with variables $\{X_1, X_2, \ldots, X_n\}$
  local variables: *queue*, a queue of arcs, initially all the arcs in *csp*

  while *queue* is not empty **do**
    $(X_i, X_j) \leftarrow$ REMOVE-FIRST(*queue*)
    if REMOVE-INCONSISTENT-VALUES(X_i, X_j) **then**
      for each X_k **in** NEIGHBORS[X_i] **do**
        add (X_k, X_i) to *queue*

---

function REMOVE-INCONSISTENT-VALUES(X_i, X_j) **returns** true iff succeeds
  removed ← *false*
  for each x **in** DOMAIN[X_i] **do**
    if no value y in DOMAIN[X_j] allows (x,y) to satisfy the constraint $X_i \leftrightarrow X_j$
      then delete x from DOMAIN[X_i]; *removed* ← *true*
  return *removed*

- An arc is added after a removal of value at a node
- n node in total, each has $\leq d$ values
- Total times of removal: $O(nd)$

Image credit: aima.cs.berkeley.edu

Complexity of a single run of AC-3

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   return removed
```

- An arc is added after a removal of value at a node
- n node in total, each has $\leq d$ values
- Total times of removal: $O(nd)$
- After a removal, $\leq n$ arcs added
- Total times of adding arcs: $O(n^2 d)$

Image credit: aima.cs.berkeley.edu

Complexity of a single run of AC-3

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```
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   for each x in DOMAIN[Xi] do
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         then delete x from DOMAIN[Xi]; removed ← true
   return removed
```

- An arc is added after a removal of value at a node
- n node in total, each has $\leq d$ values
- Total times of removal: $O(nd)$
- After a removal, $\leq n$ arcs added
- Total times of adding arcs: $O(n^2 d)$

- Check arc consistency per arc: $O(d^2)$

Complexity of a single run of AC-3 is at most $O(n^2 d^3)$

(Not required) Zhang&Yap (2001) show that its complexity is $O(n^2 d^2)$

Image credit: aima.cs.berkeley.edu

Backtracking Search with AC-3

function BACKTRACKING-SEARCH(*csp*) **returns** solution/failure
    return RECURSIVE-BACKTRACKING({ }, *csp*)

function RECURSIVE-BACKTRACKING(*assignment*, *csp*) **returns** soln/failure
    if *assignment* is complete **then return** *assignment*
    var ← SELECT-UNASSIGNED-VARIABLE(VARIABLES[*csp*], *assignment*, *csp*)
    for each *value* **in** ORDER-DOMAIN-VALUES(*var*, *assignment*, *csp*) **do**
        if *value* is consistent with *assignment* given CONSTRAINTS[*csp*] **then**
            add {*var* = *value*} to *assignment*
            AC-3(*csp*)
            result ← RECURSIVE-BACKTRACKING(*assignment*, ~~*csp*~~)
            if *result* ≠ *failure* **then return** *result*
            remove {*var* = *value*} from *assignment*
    return *failure*

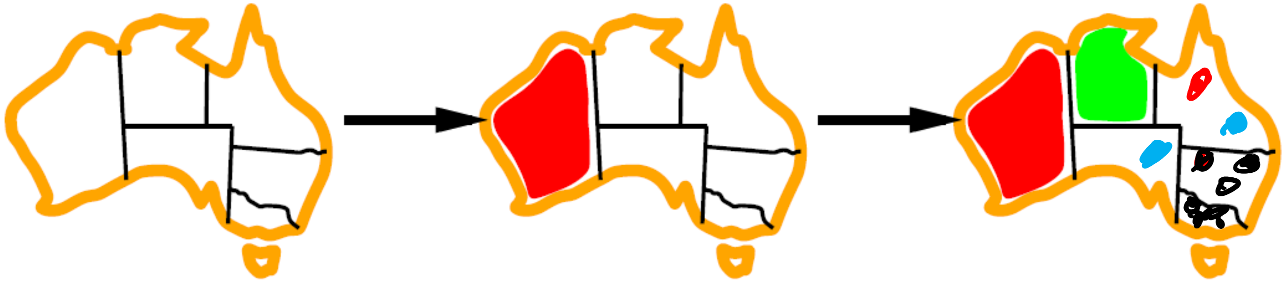
- Where do you run AC-3?

Image credit: aima.cs.berkeley.edu

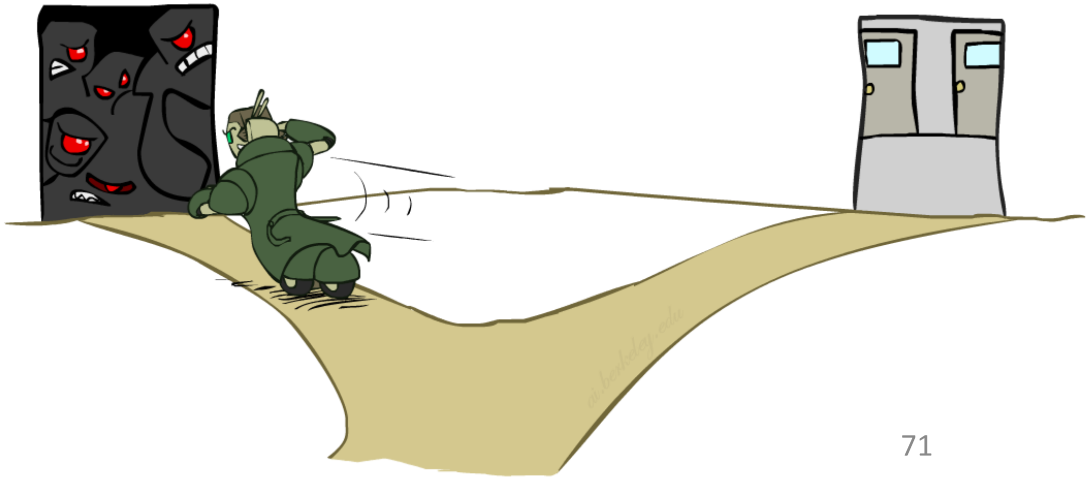
Ordering

Ordering: Minimum Remaining Values

- Variable Ordering: Minimum remaining values (MRV):
  - Choose the variable with the fewest legal left values in its domain



- Why min rather than max?
- Also called "most constrained variable"
- "Fail-fast" ordering

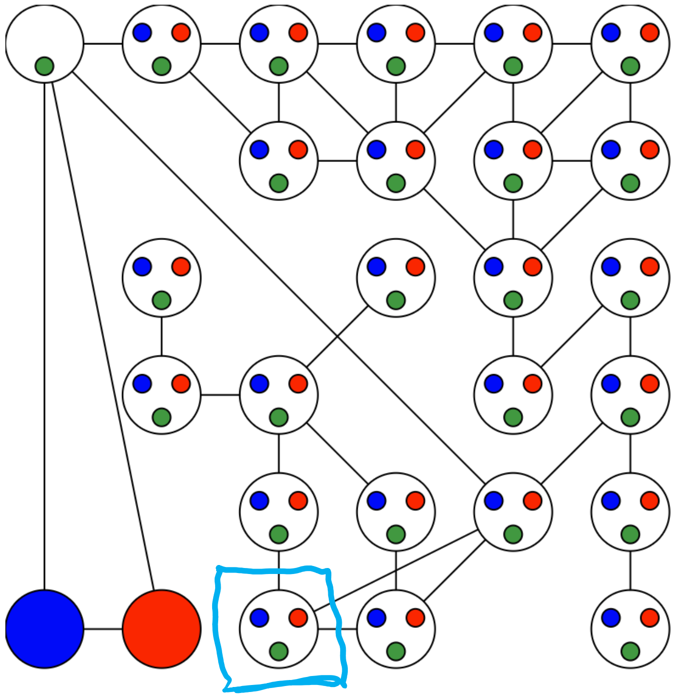


Slide/Image credit: Ketrina Yim, UCB CS 188

Demo – Coloring with a Complex Graph

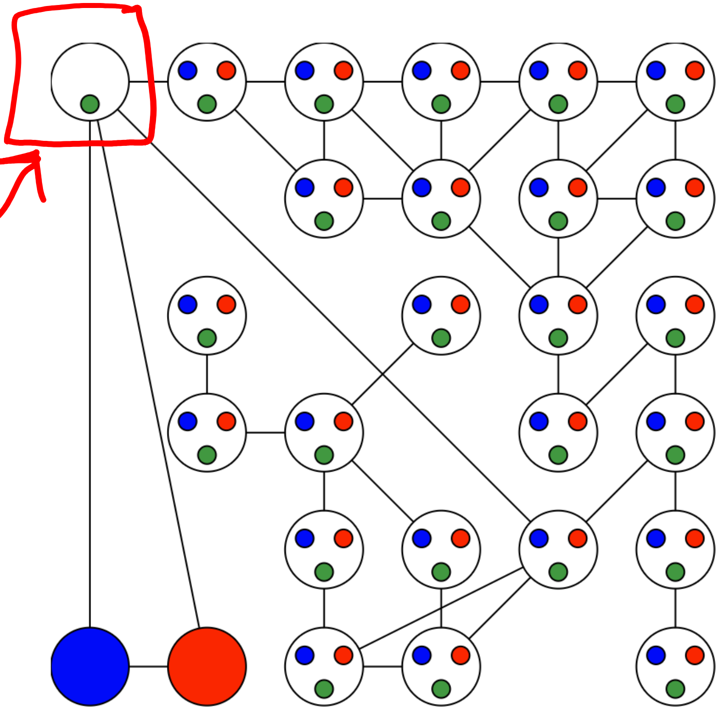
Compare

- Backtracking with Forward Checking
- Backtracking with AC-3
- Backtracking + Forward Checking + Minimum Remaining Values (MRV)



next variable in default ordering

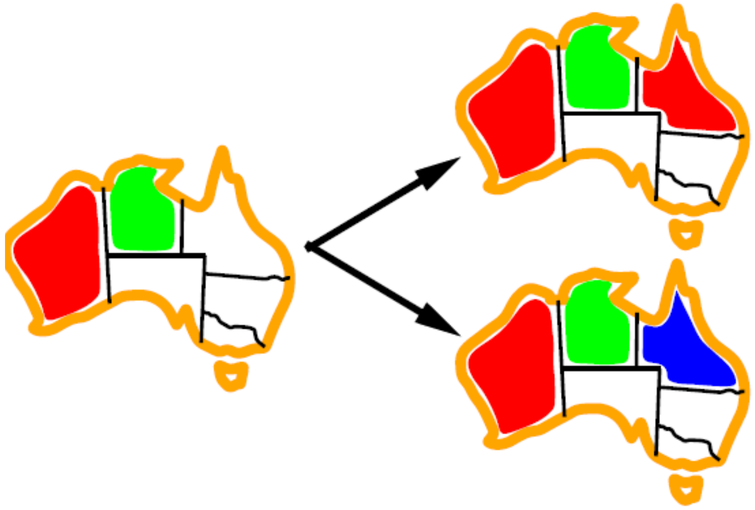
next variable with MRV

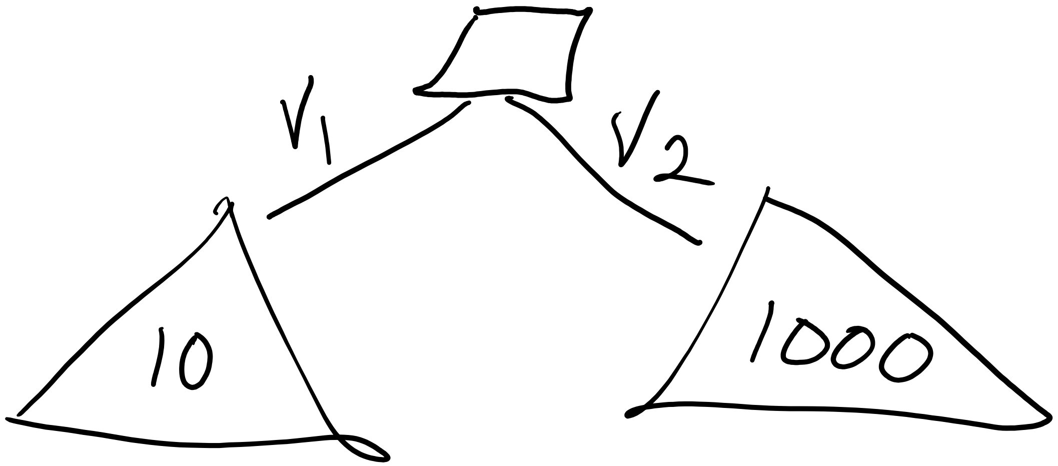


Demo credit: UCB CS 188

Ordering: Least Constraining Value

- Value Ordering: Least Constraining Value
  - Given a choice of variable, choose the *least constraining value*
  - i.e., the one that rules out the fewest values in the remaining variables
  - Note that it may take some computation to determine this! (E.g., rerunning filtering)





Ordering: Least Constraining Value

- Value Ordering: Least Constraining Value
  - Given a choice of variable, choose the *least constraining value*
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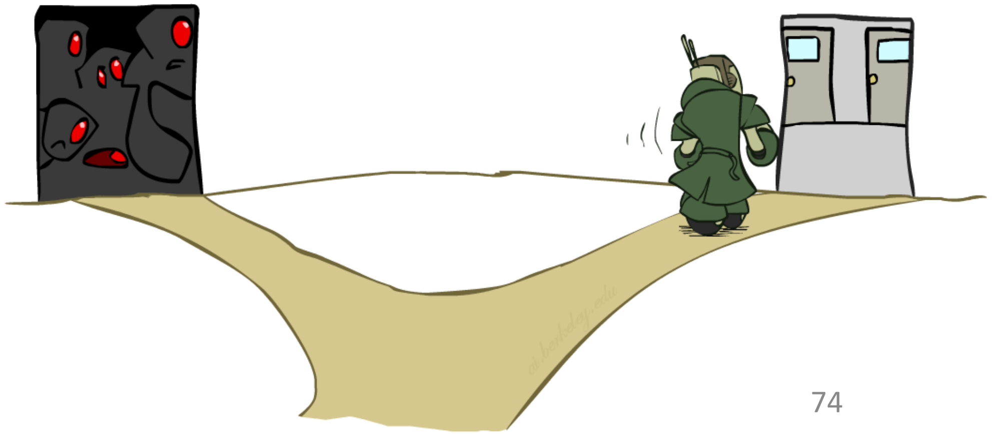
- Why least rather than most?

- Combining these ordering ideas makes 1000 queens feasible



Slide/Image credit: Ketrina Yim, UCB CS 188

Demo – Coloring with a Complex Graph

Compare

- Backtracking with Forward Checking
- Backtracking with AC-3
- Backtracking + Forward Checking + Minimum Remaining Values (MRV)
- Backtracking + AC-3 + MRV + LCV

Summary: CSPs

- CSPs are a special kind of search problem:
  - States are partial assignments
  - Goal test defined by constraints
- Basic solution: backtracking search
- Speed-ups:
  - Ordering
  - Filtering
  - (Structure)