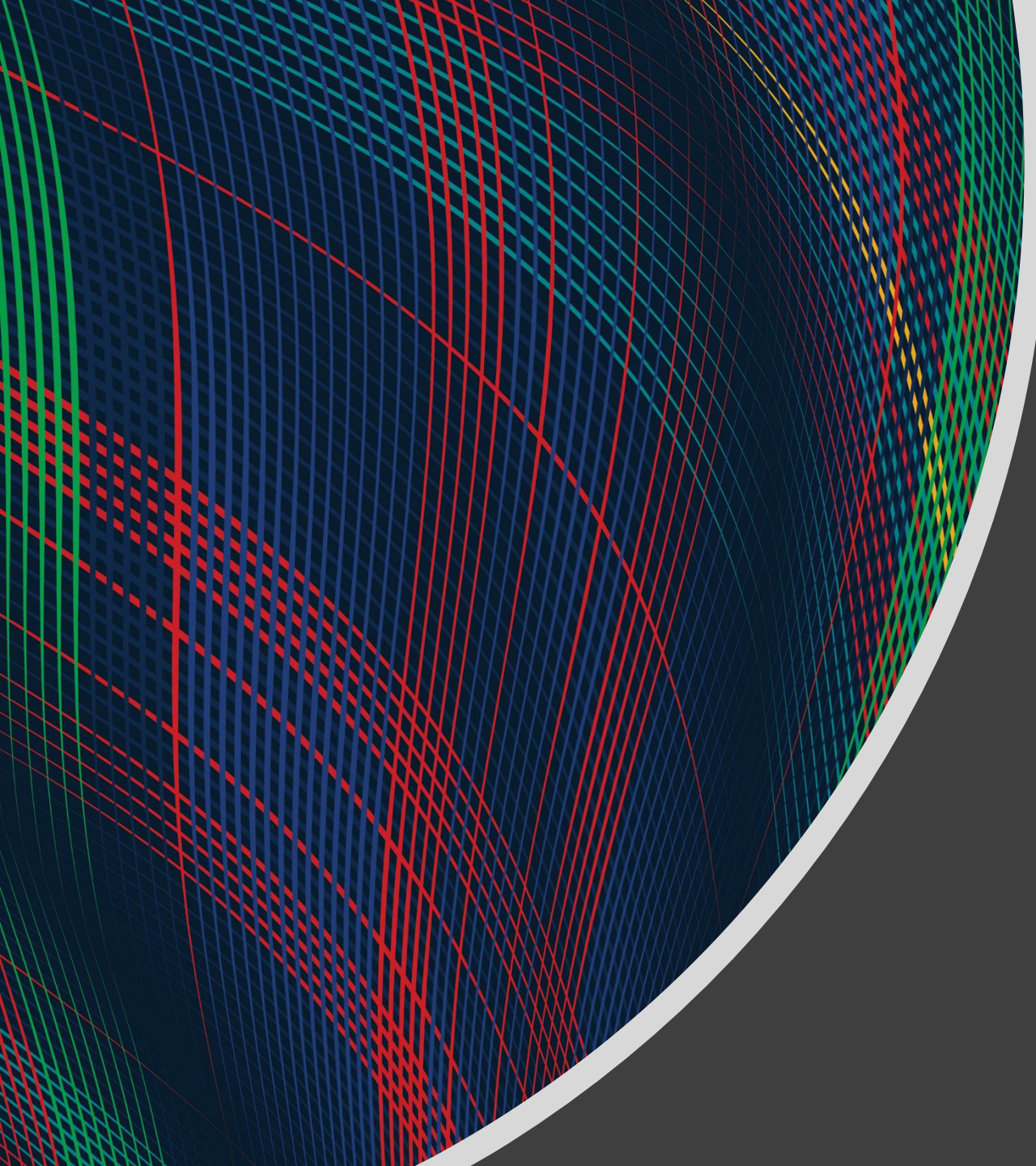


An abstract graphic on the left side of the slide, featuring a sphere-like shape composed of a dense grid of intersecting red, green, and blue lines. The lines are curved and follow the contour of the sphere, creating a complex, woven pattern. The sphere is set against a dark gray background.

07-280
Introduction to ML

Neural Networks

Instructor:
Pat Virtue & Nihar Shah

Poll 1

Logistic regression for $28 \times 28 = 784$ pixel hand-written digit images into 10 classes:

How many parameters (including bias terms)?

- A. 10
- B. $10 + 784$
- C. $10 * 784$
- D. $10 * 784 + 10$
- E. $10 * 784 + 784$
- F. I have no idea

Outline

Pre-reading: Neural Networks

- Network diagrams for linear and logistic regression
- Neuron and activation functions
- Three-neuron network
- Neural network structure (adding more neurons)

Today: Neural Networks

- Neural Network Regression
- Scalar Optimization (Backpropagation)

Next time: Neural Network

- Vector/Matrix Calculus Optimization (Backpropagation)
- Properties and Intuition

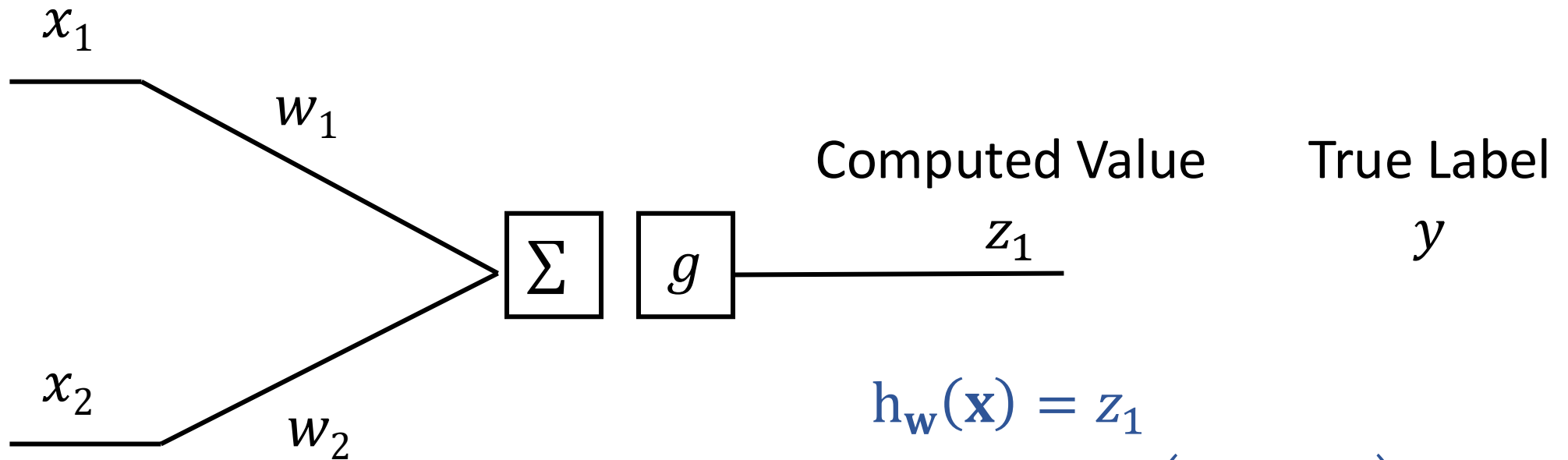
Neuron

Pre-reading

Single Neuron

Single neuron system

- Perceptron (if g is step function)
- Logistic regression (if g is sigmoid)
- Linear regression (if g is nothing)



$$h_{\mathbf{w}}(\mathbf{x}) = z_1$$

$$h_{\mathbf{w}}(\mathbf{x}) = g\left(\sum_i w_i x_i\right)$$

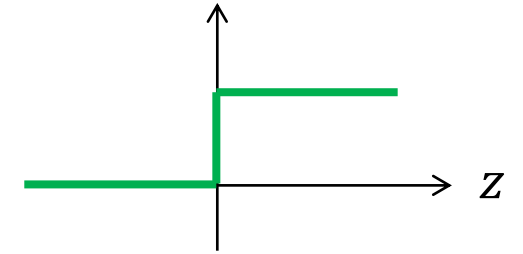
Activation Functions

Pre-reading

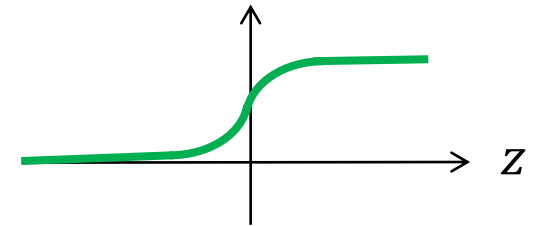
Activation Functions

It would be really helpful to have a $g(z)$ that was nicely differentiable

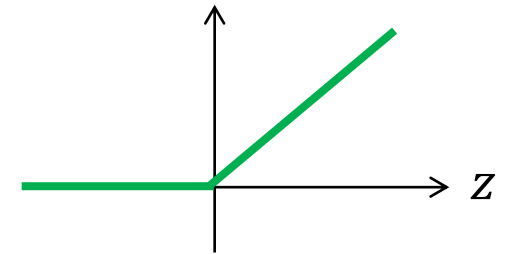
- Hard threshold: $g(z) = \begin{cases} 1 & z \geq 0 \\ 0 & z < 0 \end{cases} \quad \frac{dg}{dz} = \begin{cases} 0 & z \geq 0 \\ 0 & z < 0 \end{cases}$



- Sigmoid: $g(z) = \frac{1}{1+e^{-z}}$ $\frac{dg}{dz} = g(z)(1 - g(z))$
- (Softmax)



- ReLU: $g(z) = \max(0, z)$ $\frac{dg}{dz} = \begin{cases} 1 & z \geq 0 \\ 0 & z < 0 \end{cases}$

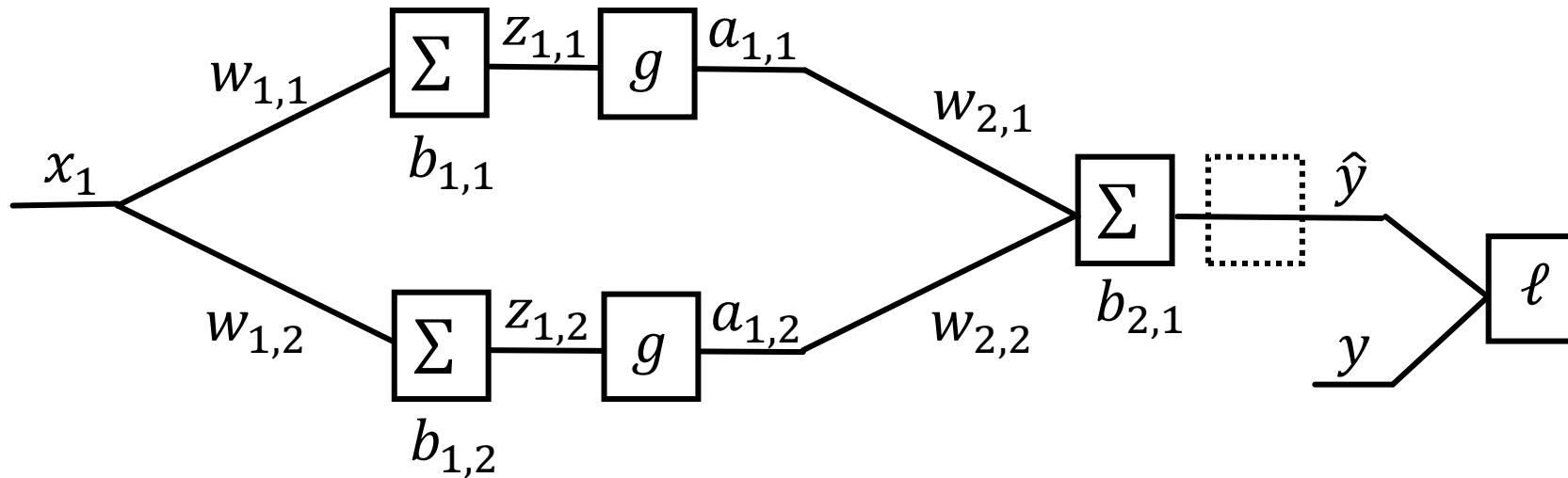


Three-neuron Network

Pre-reading

Three-neuron Network

Network diagram



$$\hat{y} = h_{\theta}(\mathbf{x}) = b_2 + \sum_j w_{2,j} a_{1,j}$$

$$a_{1,j} = g_{ReLU}(b_{1,j} + w_{1,j} x_1)$$

Adding More Neurons

Pre-reading

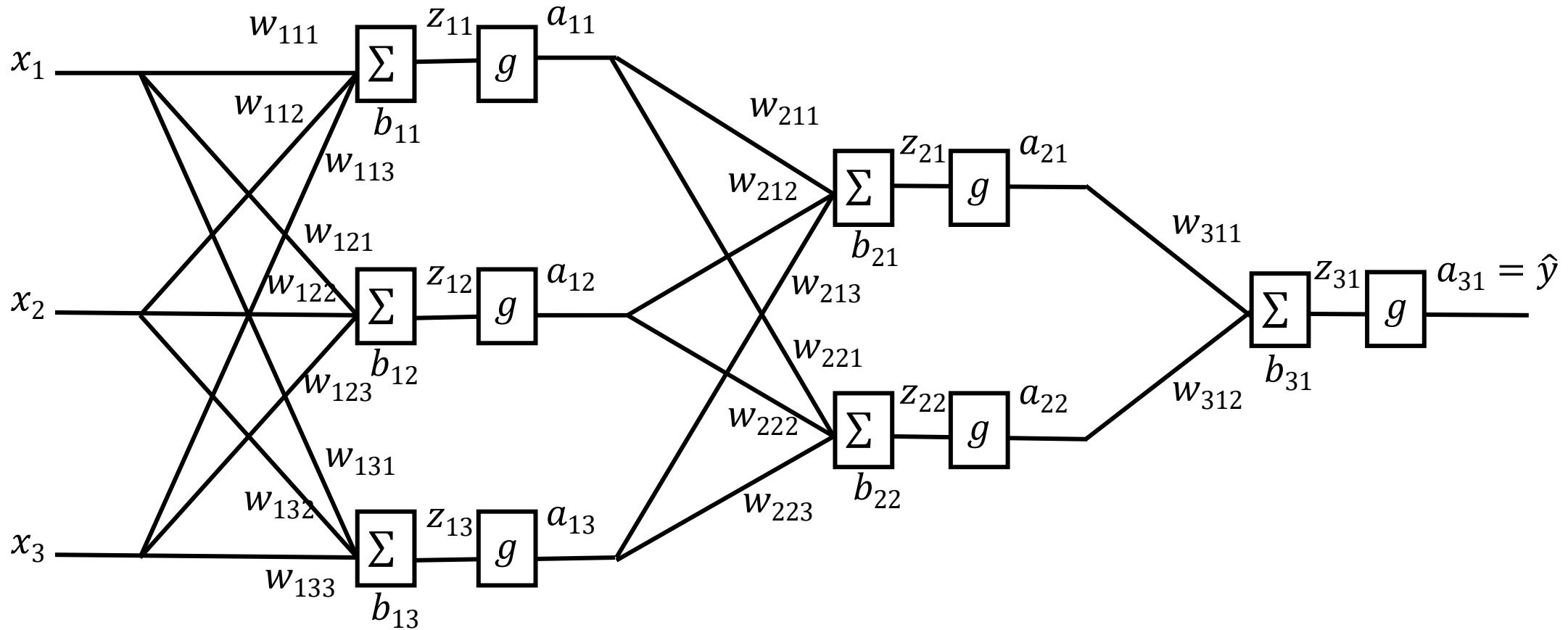
Three-layer Network

Parameter naming conventions

$w_{\text{layer output input}}$

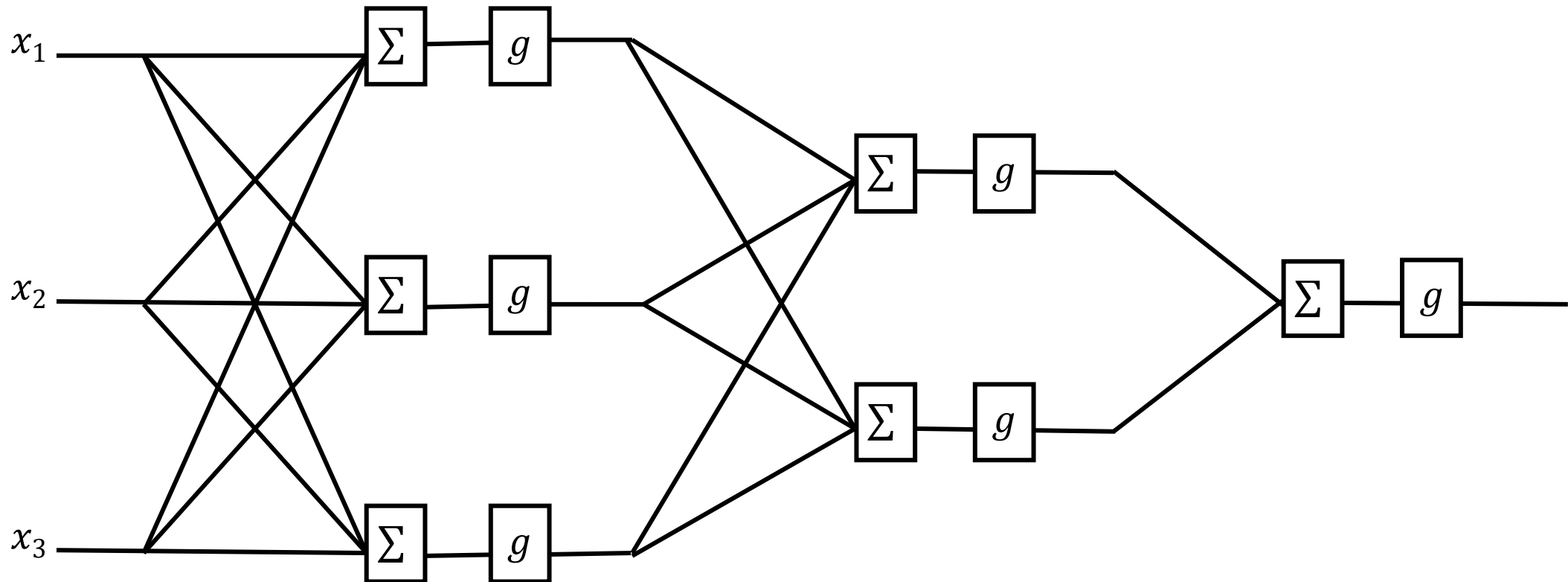
$b_{\text{layer output}}$

Network diagram



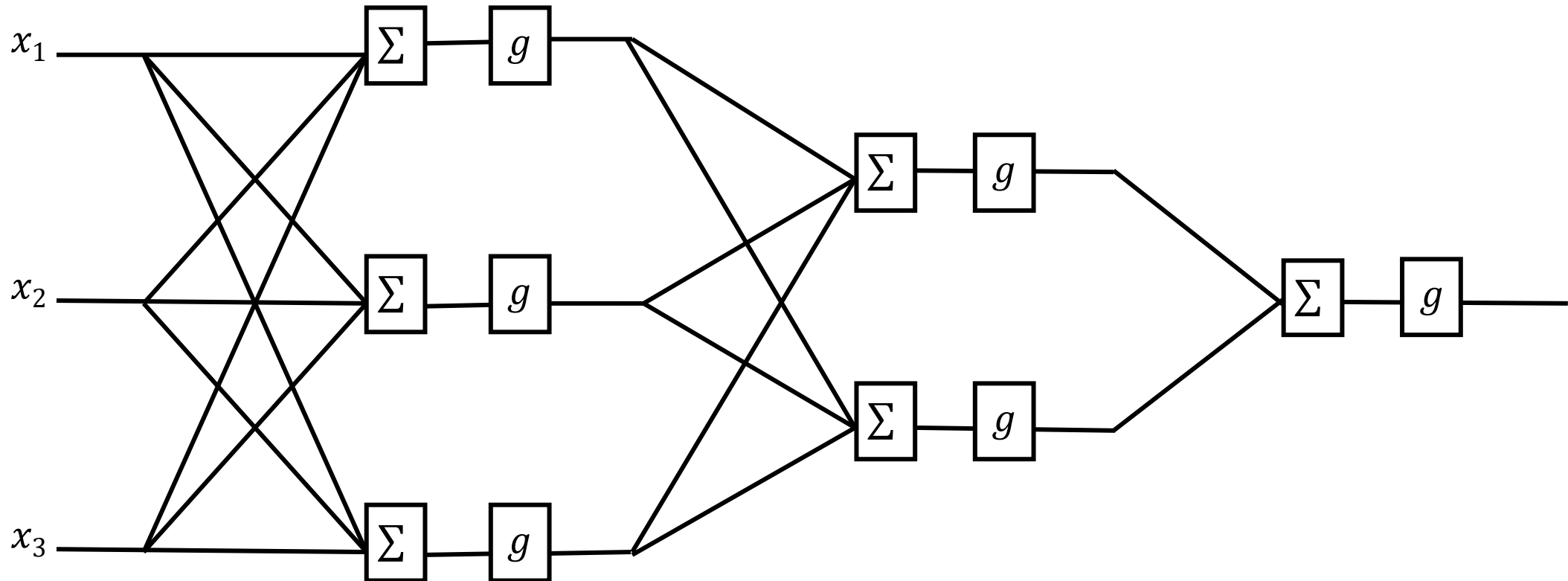
Three-layer Network

Different ways to define network layers: *Layers of functions*



Three-layer Network

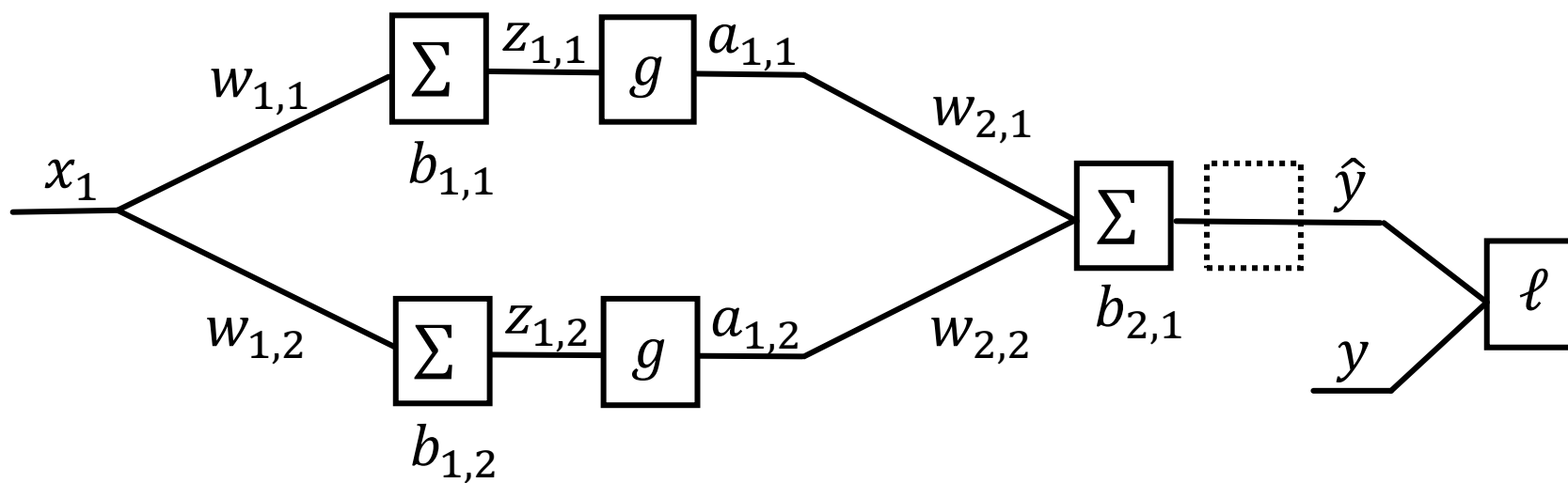
Different ways to define network layers: *Layers of neurons*



Neural Network Regression

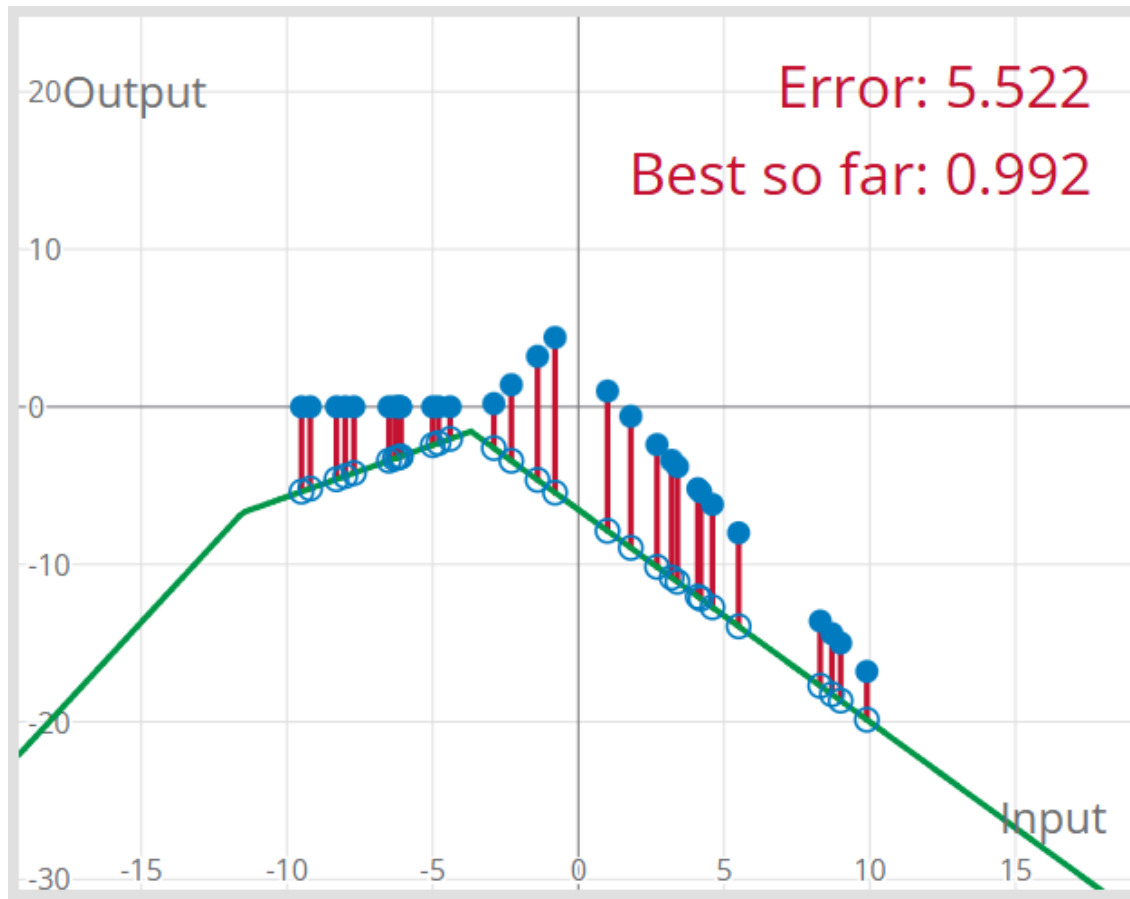
Poll 2

How many total parameters?



Three-neuron network

Interactive demo: <https://www.cs.cmu.edu/~pvirtue/ALS/regression/>

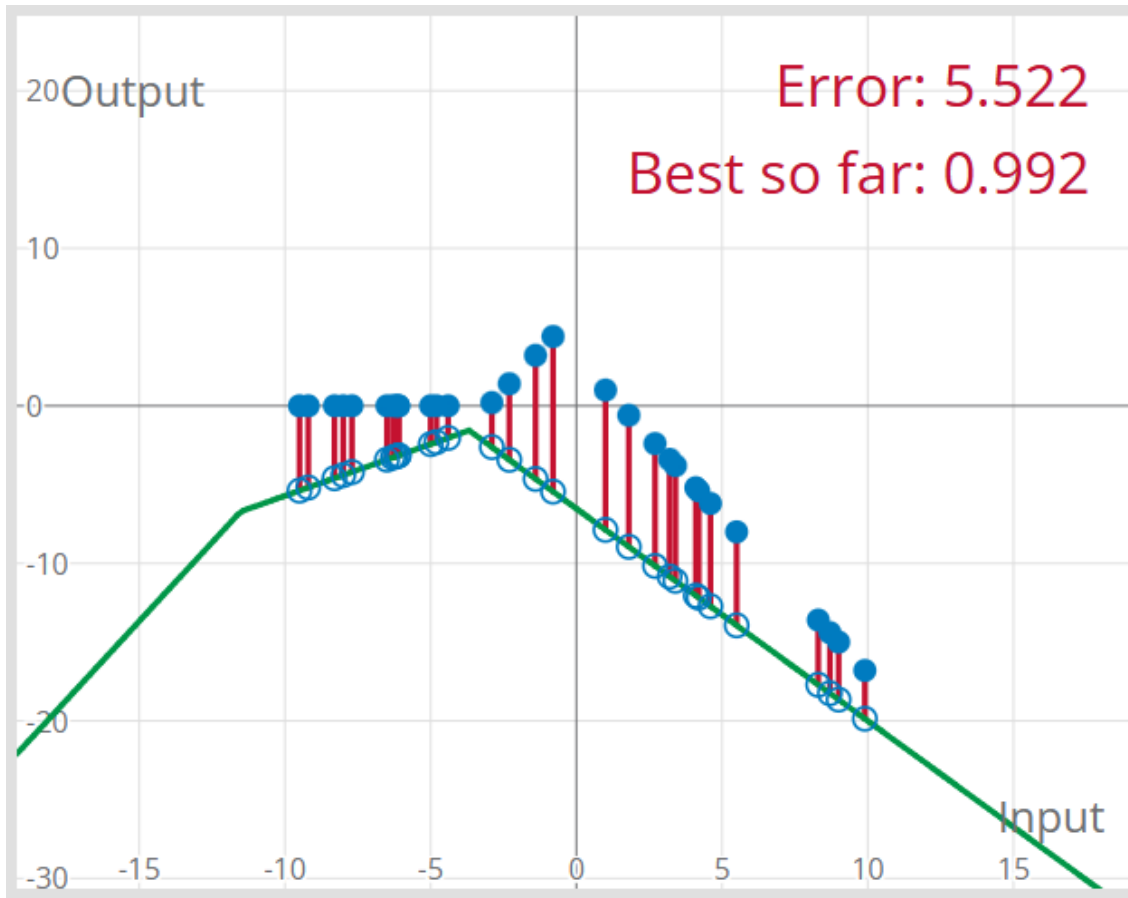


b1:	<input type="range" value="-2.72"/>	-2.72
w1:	<input type="range" value="-0.74"/>	-0.74
b2:	<input type="range" value="4.75"/>	4.75
w2:	<input type="range" value="0.41"/>	0.41
b3:	<input type="range" value="9.00"/>	9.00
w31:	<input type="range" value="-2.72"/>	-2.72
w32:	<input type="range" value="-3.27"/>	-3.27

Poll 3

Interactive demo: <https://www.cs.cmu.edu/~pvirtue/ALS/regression/>

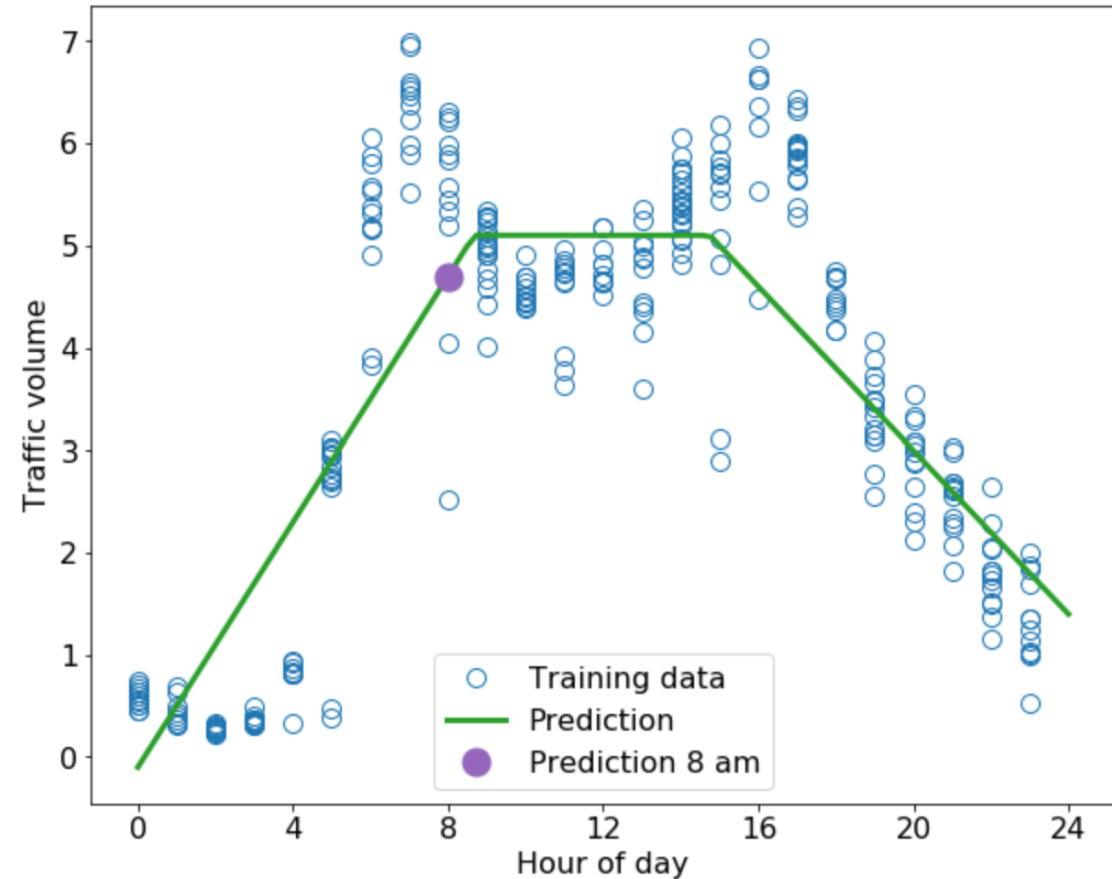
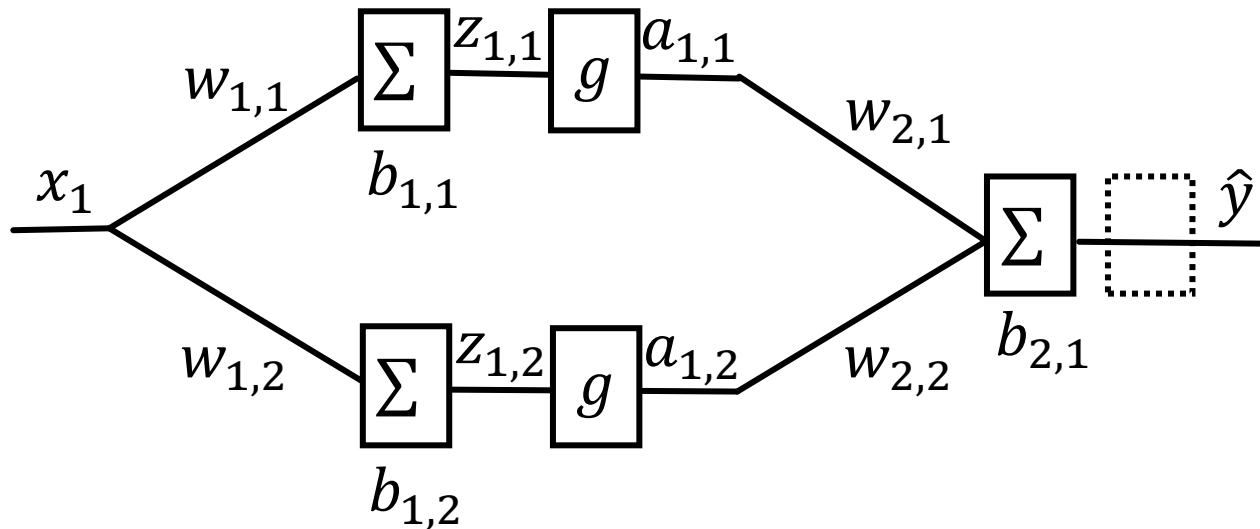
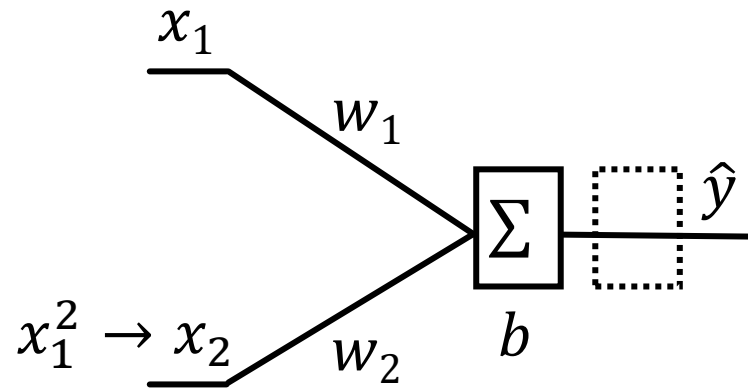
What's the smallest MSE that you can find?



b1:	<input type="range" value="-2.72"/>	-2.72
w1:	<input type="range" value="-0.74"/>	-0.74
b2:	<input type="range" value="4.75"/>	4.75
w2:	<input type="range" value="0.41"/>	0.41
b3:	<input type="range" value="9.00"/>	9.00
w31:	<input type="range" value="-2.72"/>	-2.72
w32:	<input type="range" value="-3.27"/>	-3.27

Traffic Data Example

Quadratic feature engineering vs three-neuron network



Traffic Data Example

Three-neuron network

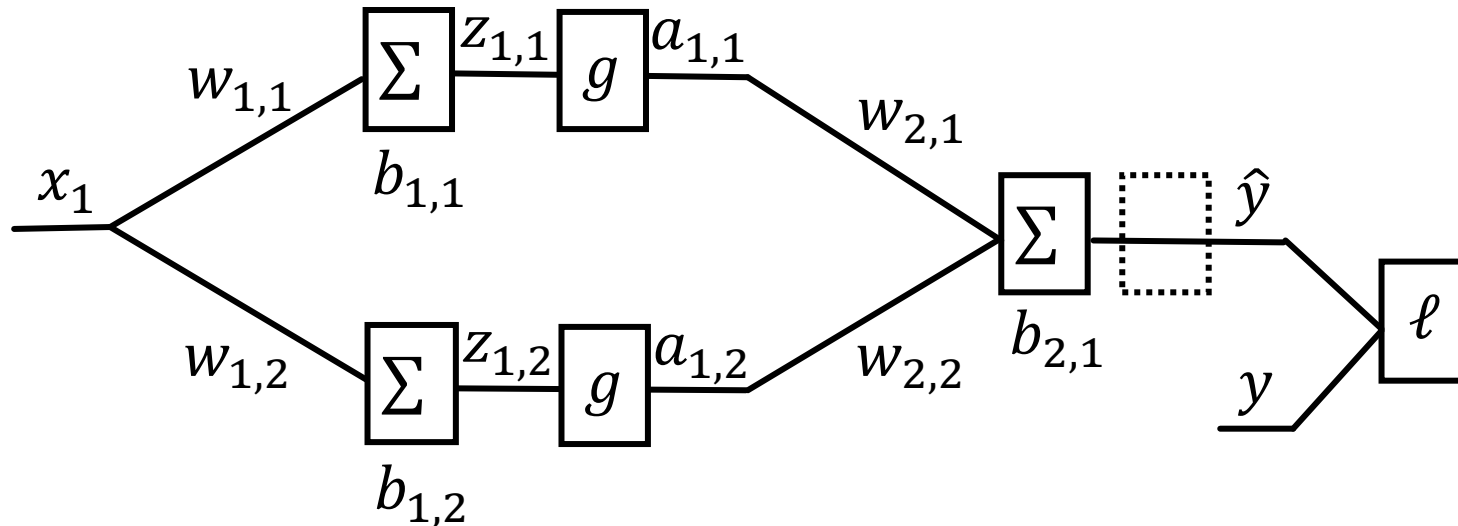
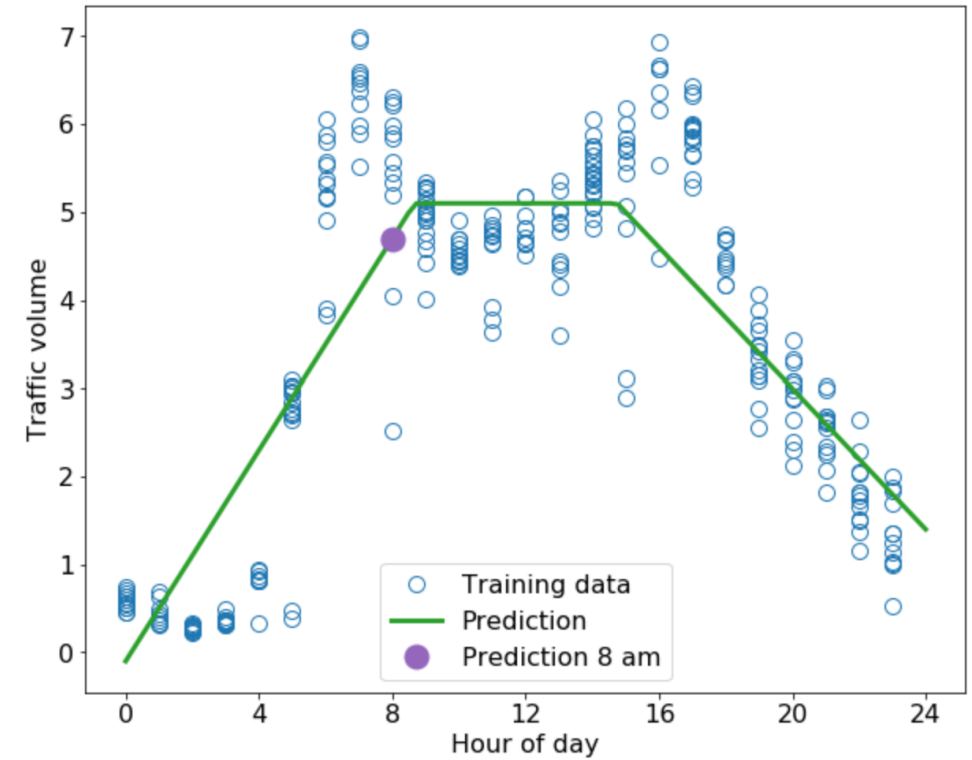
Parameters

$$w_{1,1}: 0.4 \quad b_{1,1}: -5.9$$

$$w_{1,2}: -0.6 \quad b_{1,2}: 5.2$$

$$w_{2,1}: -1.0 \quad b_{2,1}: 5.1$$

$$w_{2,2}: -1.0$$



Traffic Data Example

Three-neuron network

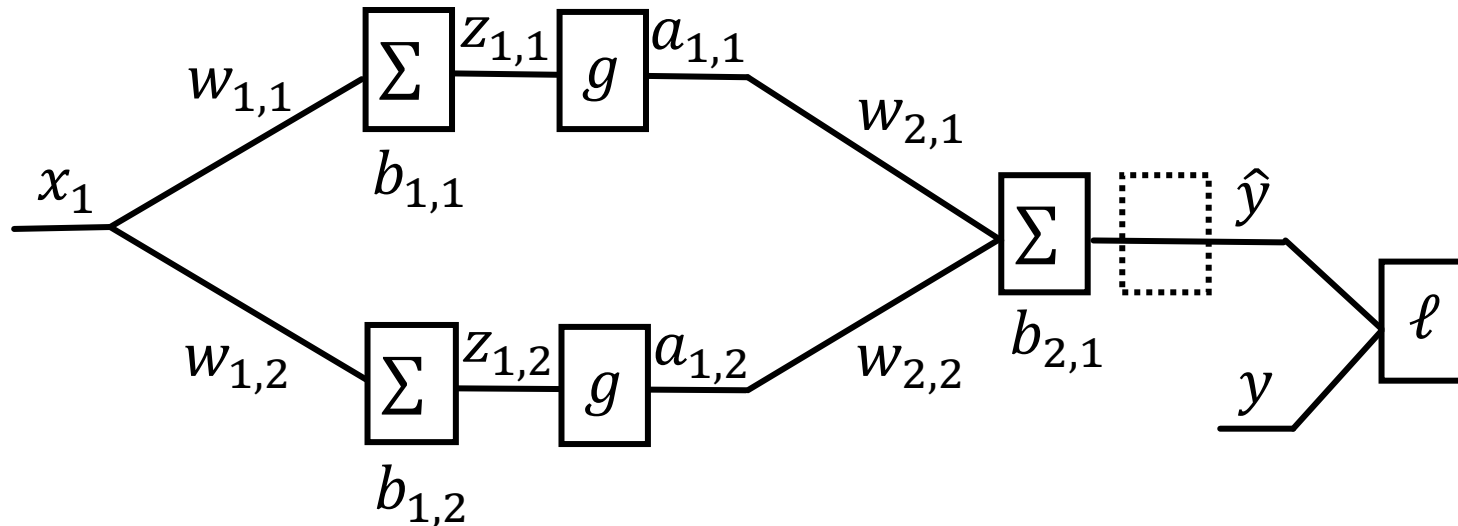
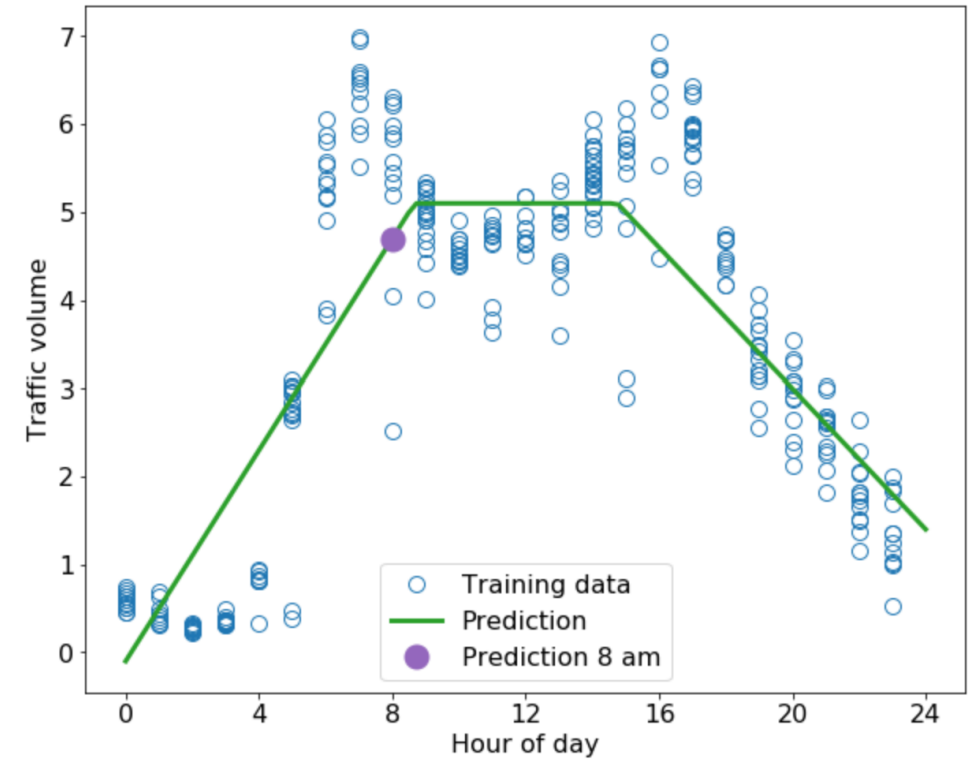
Parameters

$$w_{1,1}: 0.4 \quad b_{1,1}: -5.9$$

$$w_{1,2}: -0.6 \quad b_{1,2}: 5.2$$

$$w_{2,1}: -1.0 \quad b_{2,1}: 5.1$$

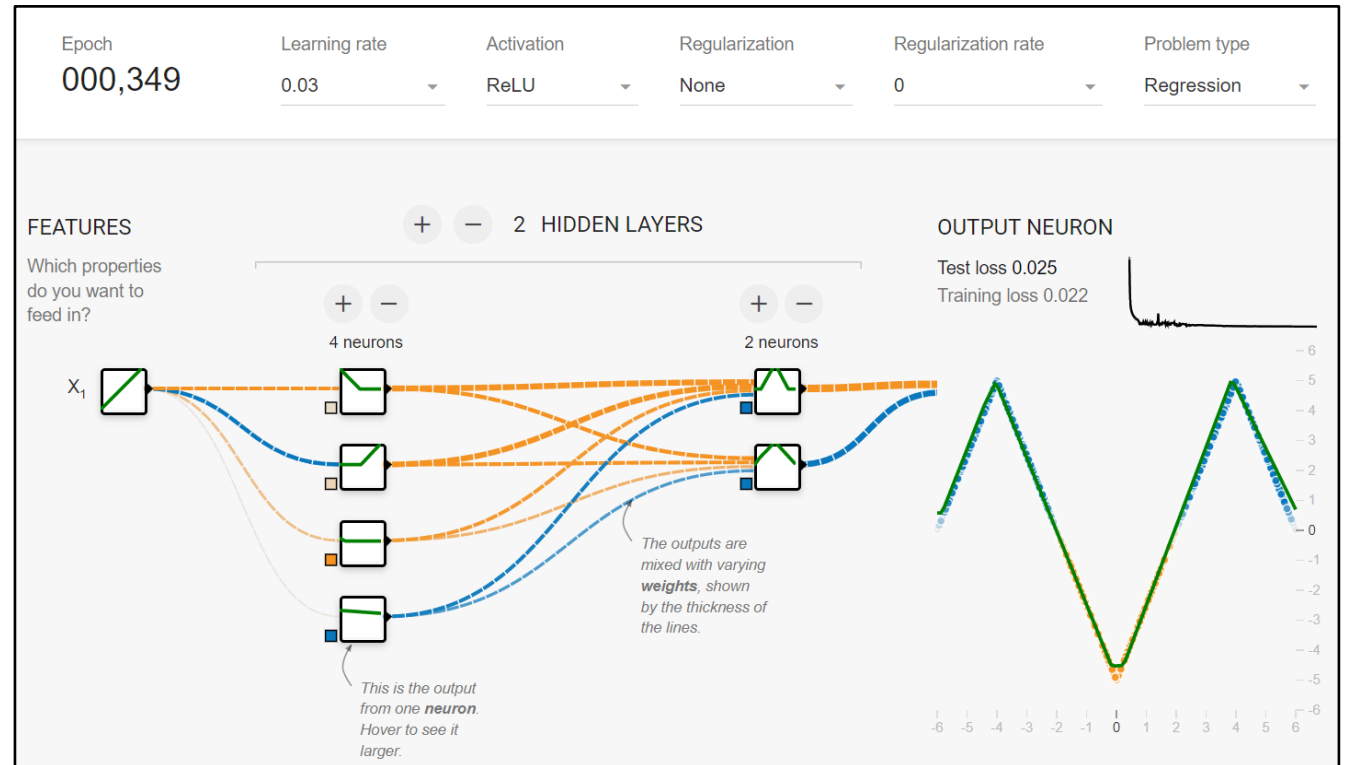
$$w_{2,2}: -1.0$$



Exercise: Neural Net Regression: 1-D input

What is the smallest number of nodes you can use to perfectly fit the training data for the "M" dataset?

[Tensor Flow Playground: 1-D Regression](https://playground.tensorflow.org/)



<https://playground.tensorflow.org/>

<https://www.cs.cmu.edu/~pvirtue/tfp>

Neural Network Optimization

Objective, Loss Functions, Gradient Descent

Objective

$$J(\theta) = \frac{1}{N} \sum_{i=1}^N J^{(i)}(\theta) \quad J^{(i)}(\theta) = \ell(y^{(i)}, \hat{y}^{(i)}) \quad \hat{y} = h(x; \theta)$$

Loss functions

- Regression $\rightarrow$ Squared error: $\ell(y, \hat{y}) = (y - \hat{y})^2$
- Classification $\rightarrow$ Cross entropy: $\ell(\mathbf{y}, \hat{\mathbf{y}}) = -\sum_k y_k \log \hat{y}_k$

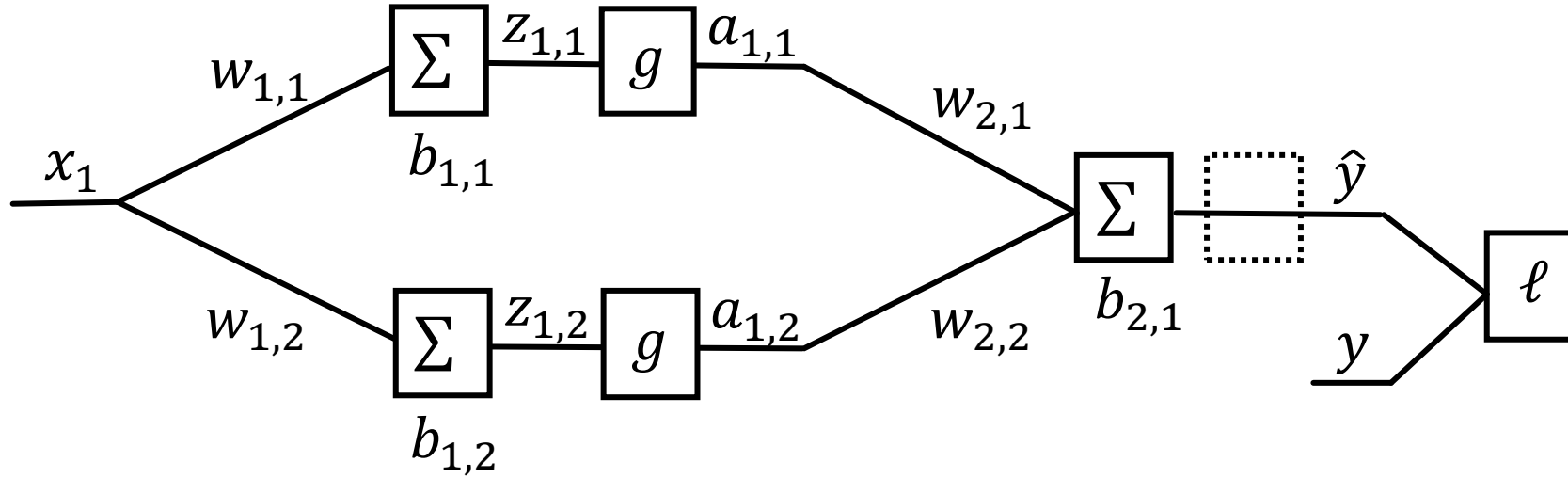
Gradient descent

while not converged

$$\theta \leftarrow \theta - \alpha \partial J / \partial \theta$$

Reminder: Poll 2

How many total parameters?



Poll 4

Which of the following updates will we execute during gradient descent?

Select all that apply

A. $x \leftarrow x - \alpha \frac{\partial J}{\partial x}$

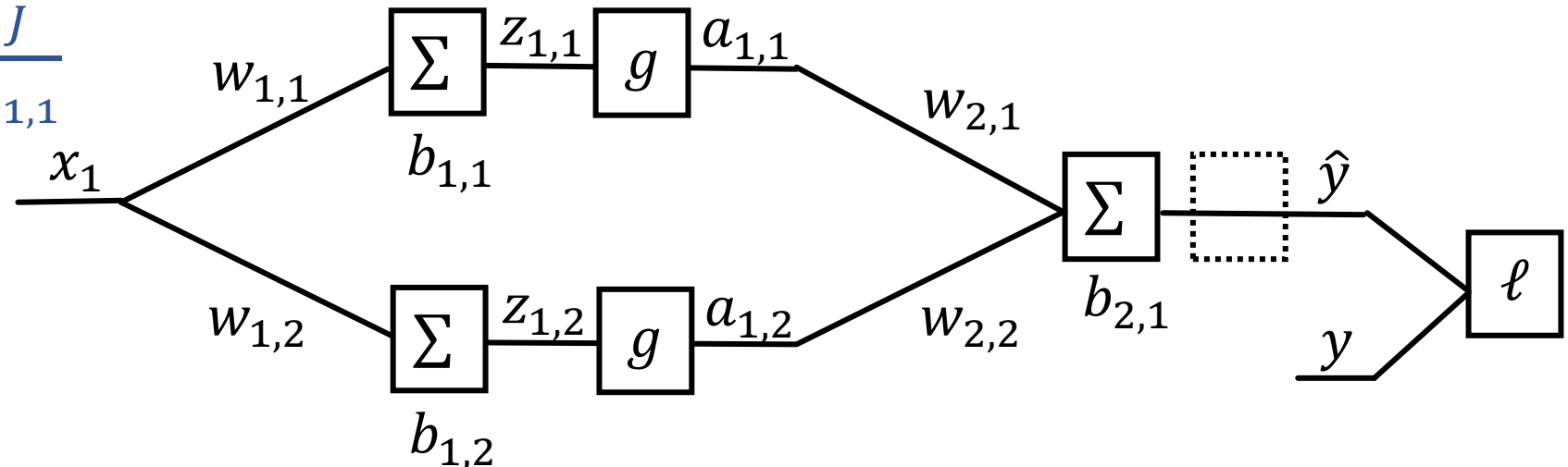
B. $w_{1,1} \leftarrow w_{1,1} - \alpha \frac{\partial J}{\partial w_{1,1}}$

C. $b_{1,1} \leftarrow b_{1,1} - \alpha \frac{\partial J}{\partial b_{1,1}}$

D. $a_{1,1} \leftarrow a_{1,1} - \alpha \frac{\partial J}{\partial a_{1,1}}$

E. $\hat{y} \leftarrow \hat{y} - \alpha \frac{\partial J}{\partial \hat{y}}$

F. $y \leftarrow y - \alpha \frac{\partial J}{\partial y}$

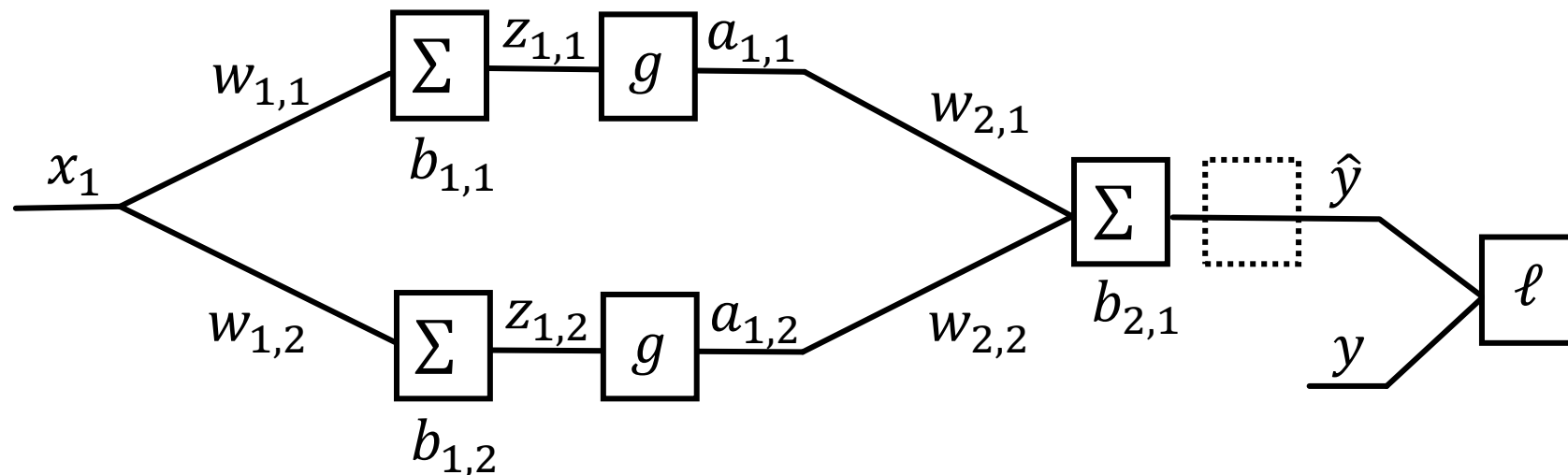


Optimization

Three-neuron network

$$\hat{y} = h_{\theta}(\mathbf{x}) = b_2 + \sum_j w_{2,j} a_{1,j}$$

$$a_{1,j} = g(b_{1,j} + w_{1,j} x_1)$$



$$\frac{\partial J}{\partial w_{2,1}} =$$

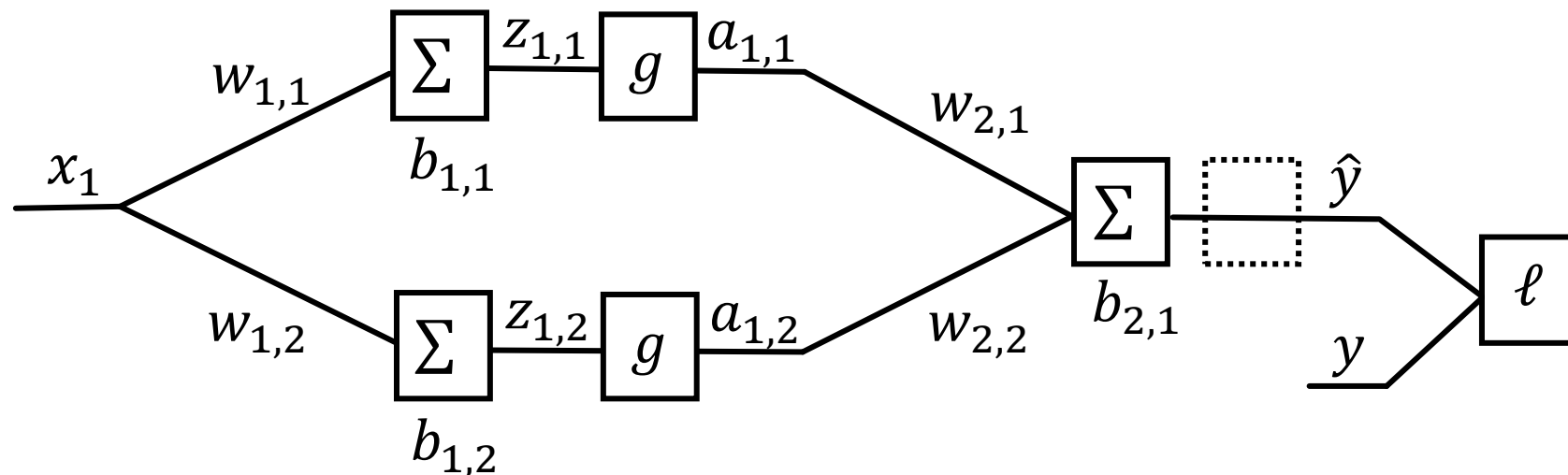
$$\frac{\partial J}{\partial b_{2,1}} =$$

Optimization

Three-neuron network

$$\hat{y} = h_{\theta}(\mathbf{x}) = b_2 + \sum_j w_{2,j} a_{1,j}$$

$$a_{1,j} = g(b_{1,j} + w_{1,j} x_1)$$



$$\frac{\partial J}{\partial w_{1,1}} =$$

$$\frac{\partial J}{\partial b_{1,1}} =$$

$$\frac{\partial J}{\partial w_{2,1}} = \frac{\partial \ell}{\partial \hat{y}} \frac{\partial \hat{y}}{\partial w_{2,1}}$$

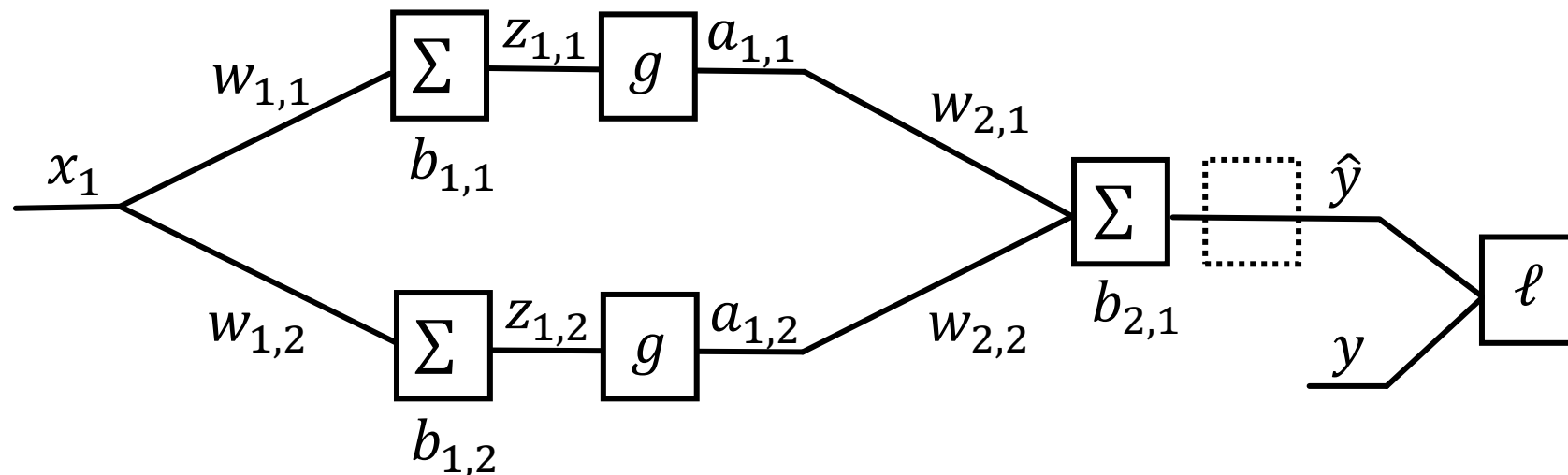
$$\frac{\partial J}{\partial b_{2,1}} = \frac{\partial \ell}{\partial \hat{y}} \frac{\partial \hat{y}}{\partial b_{2,1}}$$

Optimization

Three-neuron network

$$\hat{y} = h_{\theta}(\mathbf{x}) = b_2 + \sum_j w_{2,j} a_{1,j}$$

$$a_{1,j} = g(b_{1,j} + w_{1,j} x_1)$$



$$\frac{\partial J}{\partial w_{1,j}} = \frac{\partial \ell}{\partial \hat{y}} \frac{\partial \hat{y}}{\partial a_{1,j}} \frac{\partial a_{1,j}}{\partial z_{1,j}} \frac{\partial z_{1,j}}{\partial w_{1,j}}$$

$$\frac{\partial J}{\partial b_{1,j}} = \frac{\partial \ell}{\partial \hat{y}} \frac{\partial \hat{y}}{\partial a_{1,j}} \frac{\partial a_{1,j}}{\partial z_{1,j}} \frac{\partial z_{1,j}}{\partial b_{1,j}}$$

$$\frac{\partial J}{\partial w_{2,j}} = \frac{\partial \ell}{\partial \hat{y}} \frac{\partial \hat{y}}{\partial w_{2,j}}$$

$$\frac{\partial J}{\partial b_{2,1}} = \frac{\partial \ell}{\partial \hat{y}} \frac{\partial \hat{y}}{\partial b_{2,1}}$$

Neural Net Forward Pass and Backpropagation

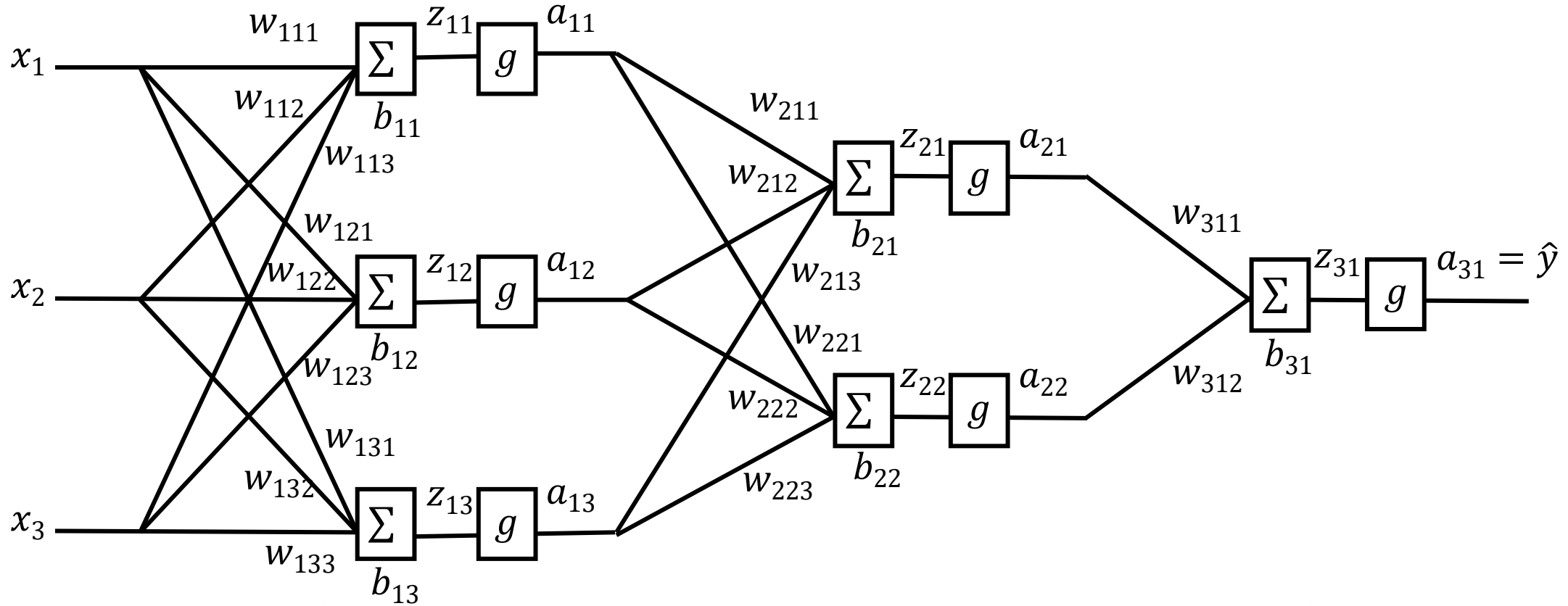
Forward pass

Parameter naming conventions

$w_{\text{layer output input}}$

$b_{\text{layer output}}$

Predicting output from input

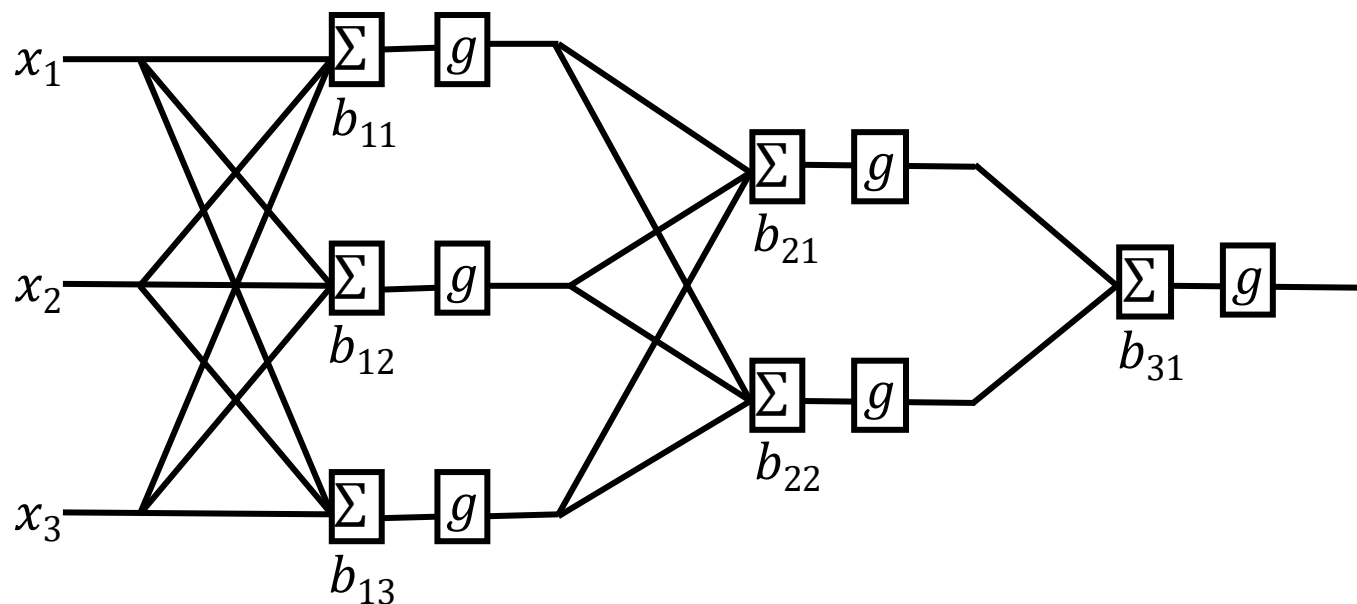


$$\hat{y} = h_{\theta}(\mathbf{x}) = g \left(b_{3,1} + \sum_k w_{3,1,k} g \left(b_{2,k} + \sum_i w_{2,k,i} g \left(b_{1,i} + \sum_j w_{1,i,j} x_j \right) \right) \right)$$

Optimization

$$\hat{y} = h_{\theta}(\mathbf{x}) = g \left(b_{3,1} + \sum_k w_{3,1,k} g \left(b_{2,k} + \sum_i w_{2,k,i} g \left(b_{1,i} + \sum_j w_{1,i,j} x_j \right) \right) \right)$$

Tons of repeated partial derivatives



$$\frac{\partial J}{\partial b_{3,1}} = \frac{\partial \ell}{\partial \hat{y}} \frac{\partial a_{3,1}}{\partial z_{3,1}} \frac{\partial z_{3,1}}{\partial b_{3,1}}$$

$$\frac{\partial J}{\partial b_{2,i}} = \frac{\partial \ell}{\partial \hat{y}} \frac{\partial a_{3,1}}{\partial z_{3,1}} \frac{\partial z_{3,1}}{\partial a_{2,i}} \frac{\partial a_{2,i}}{\partial z_{2,i}} \frac{\partial z_{2,i}}{\partial b_{2,i}}$$

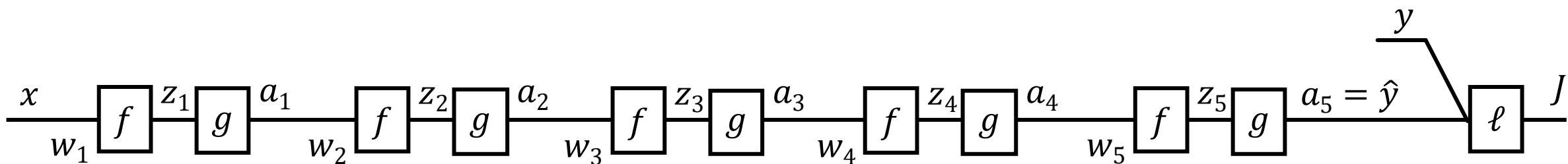
$$\frac{\partial J}{\partial b_{1,i}} = \frac{\partial \ell}{\partial \hat{y}} \frac{\partial a_{3,1}}{\partial z_{3,1}} \frac{\partial z_{3,1}}{\partial \mathbf{a}_2} \frac{\partial \mathbf{a}_2}{\partial \mathbf{z}_2} \frac{\partial \mathbf{z}_2}{\partial a_{1,i}} \frac{\partial a_{1,i}}{\partial z_{1,i}} \frac{\partial z_{1,i}}{\partial b_{1,i}}$$

Forward Pass and Backpropagation

Scalar's version

Forward Pass

Width 1 deep network (no bias) (dumb but will help with calculus)

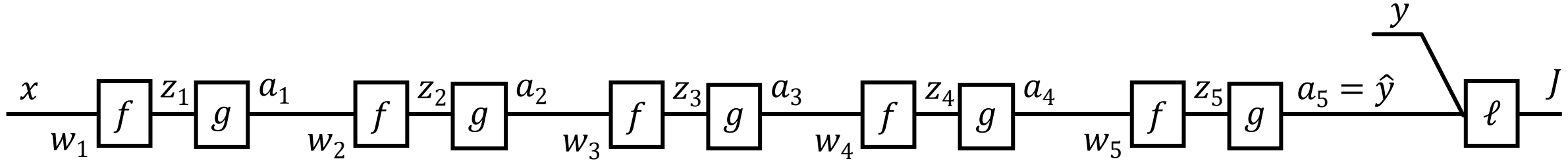


$$\begin{aligned} z_\ell &= f(w_\ell, a_{\ell-1}) = w_\ell \cdot a_{\ell-1} \\ a_\ell &= g(z_\ell) \end{aligned} \qquad \begin{aligned} \hat{y} &= h_{\boldsymbol{\theta}}(\mathbf{x}) = g\left(w_5 \cdot g\left(w_4 \cdot g\left(w_3 \cdot g\left(w_2 \cdot g(w_1 \cdot x)\right)\right)\right)\right) \\ &= g\left(f\left(g\left(f\left(g\left(f\left(g\left(f\left(g(f(x))\right)\right)\right)\right)\right)\right)\right)\right) \end{aligned}$$

Forward Pass

$$\hat{y} = h_{\theta}(\mathbf{x}) = g\left(w_5 \cdot g\left(w_4 \cdot g\left(w_3 \cdot g\left(w_2 \cdot g(w_1 \cdot x)\right)\right)\right)\right)$$

Width 1 deep network (no bias) (dumb but will help with calculus)

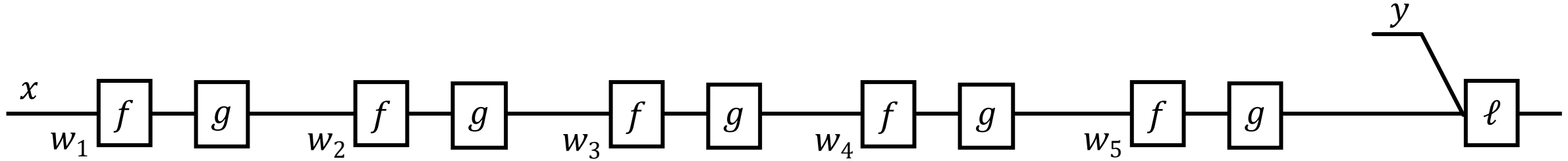


With a new data point (x, y) , we have our current weight values
but we don't $\hat{y}$ (or any of the intermediate values z_ℓ and a_ℓ)
or the value of our object function J

Forward Pass

$$\hat{y} = h_{\theta}(\mathbf{x}) = g\left(w_5 \cdot g\left(w_4 \cdot g\left(w_3 \cdot g\left(w_2 \cdot g(w_1 \cdot x)\right)\right)\right)\right)$$

Width 1 deep network (no bias) (dumb but will help with calculus)

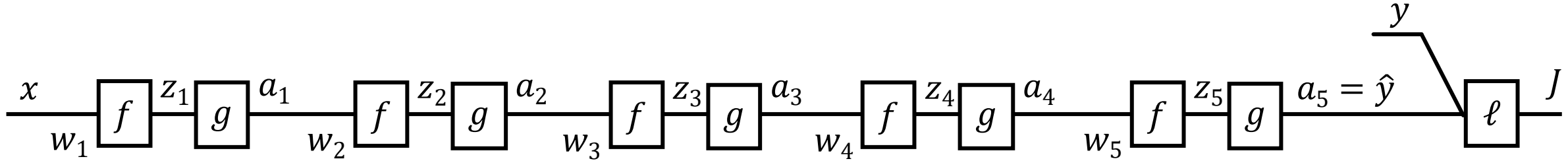


With a new data point (x, y) , we have our current weight values but we don't $\hat{y}$ (or any of the intermediate values z_ℓ and a_ℓ) or the value of our object function J

Forward Pass

$$\hat{y} = h_{\theta}(\mathbf{x}) = g\left(w_5 \cdot g\left(w_4 \cdot g\left(w_3 \cdot g\left(w_2 \cdot g(w_1 \cdot x)\right)\right)\right)\right)$$

Width 1 deep network (no bias) (dumb but will help with calculus)



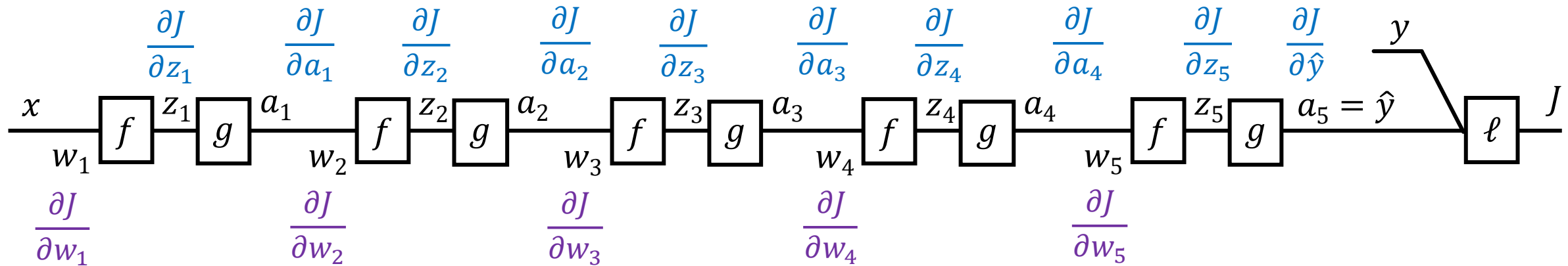
With a new data point (x, y) , we have our current weight values but we don't have $\hat{y}$ (or any of the intermediate values z_ℓ and a_ℓ) or the value of our object function J

The forward pass propagates x (forward) through the network to give us these values

Backpropagation

$$\hat{y} = h_{\theta}(\mathbf{x}) = g\left(w_5 \cdot g\left(w_4 \cdot g\left(w_3 \cdot g\left(w_2 \cdot g(w_1 \cdot x)\right)\right)\right)\right)$$

Width 1 deep network (no bias) (dumb but will help with calculus)



To do gradient descent we need the partial derivative of the objective with respect to each parameter, $w_\ell \leftarrow w_\ell - \alpha \frac{\partial J}{\partial w_\ell}$

The backward pass propagates the change in the objective with respect to intermediate values ($\frac{\partial J}{\partial z_\ell}$ and $\frac{\partial J}{\partial a_\ell}$) back through the network to produce each $\frac{\partial J}{\partial w_\ell}$

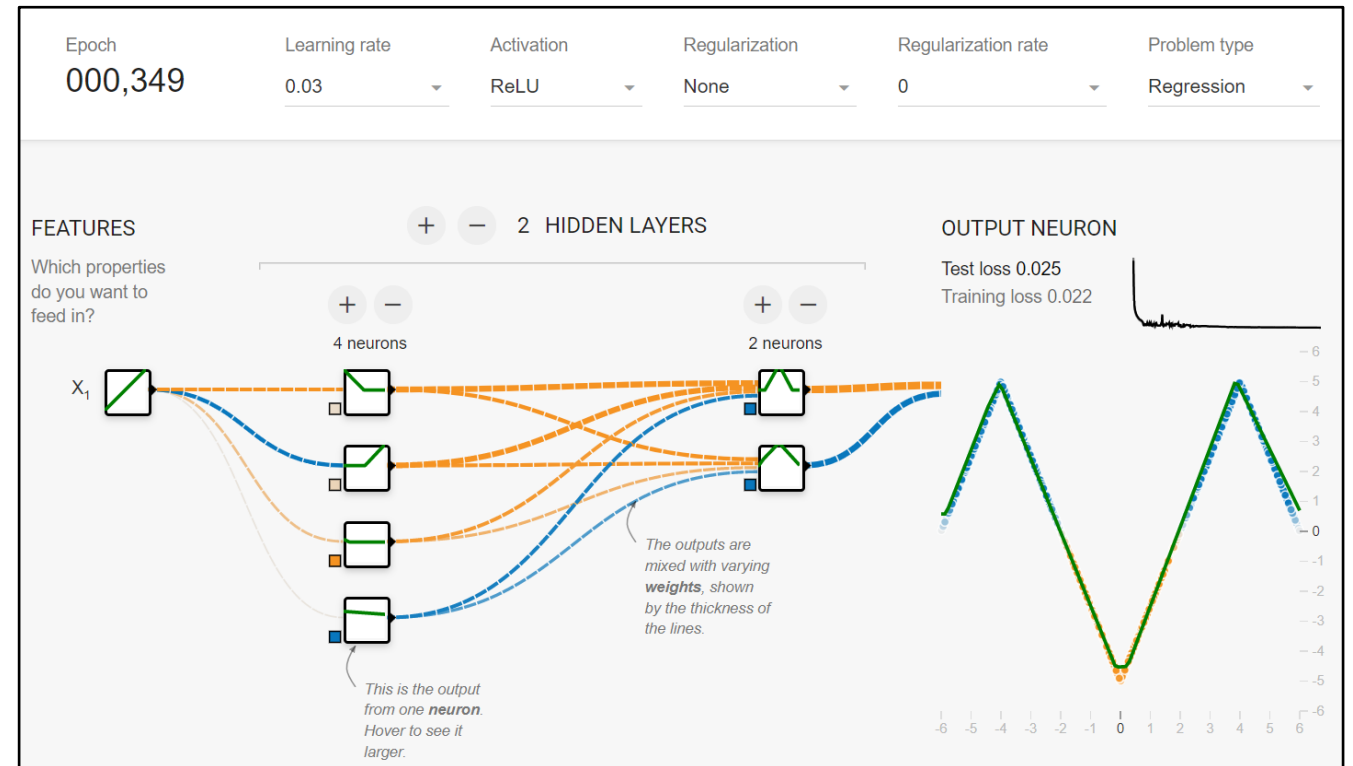
Additional Slides

Simple Network

What do the colors of the lines represent?

A. Weight

B. Value from previous neuron



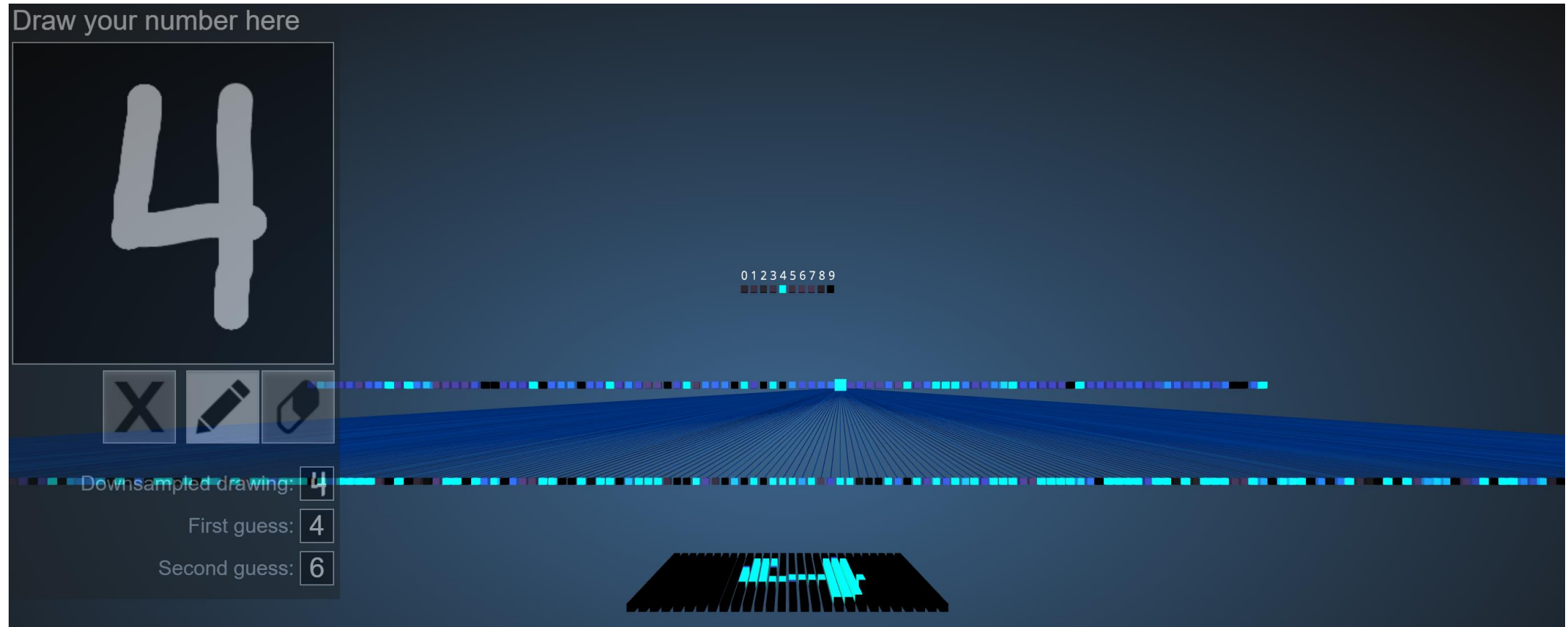
<https://playground.tensorflow.org/>

<https://www.cs.cmu.edu/~pvirtue/tfp>

Image Classification

Demo of (Fully-connected) neural network to classify images of hand-written digits

https://adamharley.com/nn_vis/mlp/3d.html



Reminder: Image Classification

Demo of (Fully-connected) neural network to classify images of hand-written digits

https://adamharley.com/nn_vis/mlp/3d.html

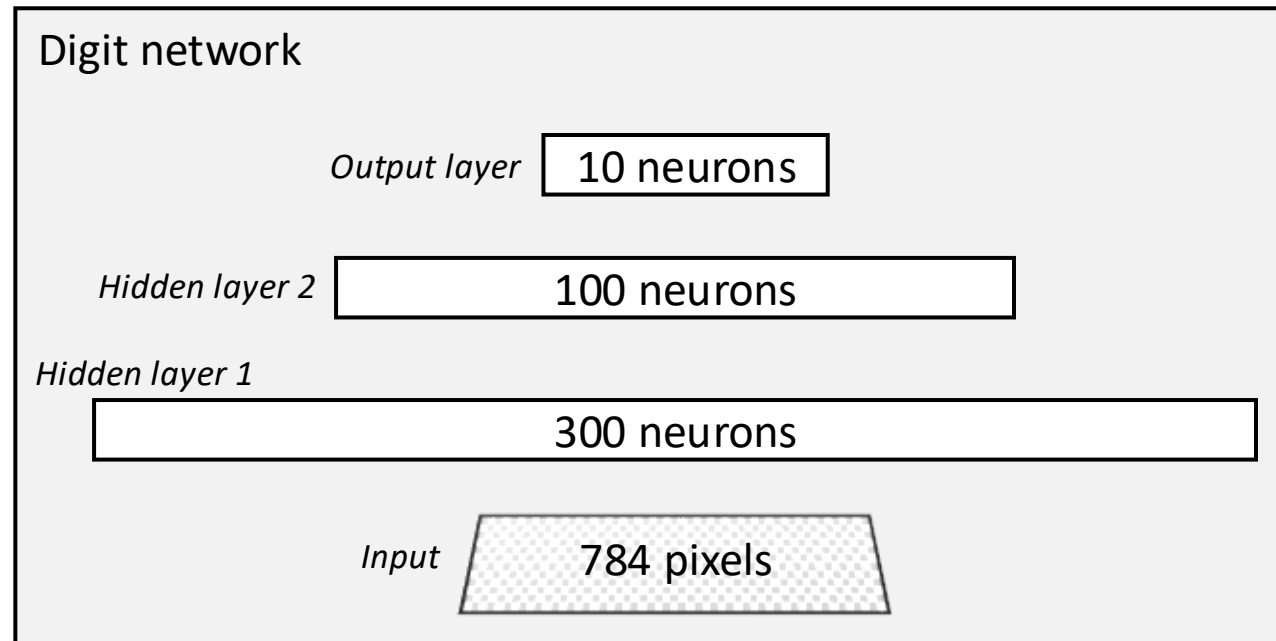
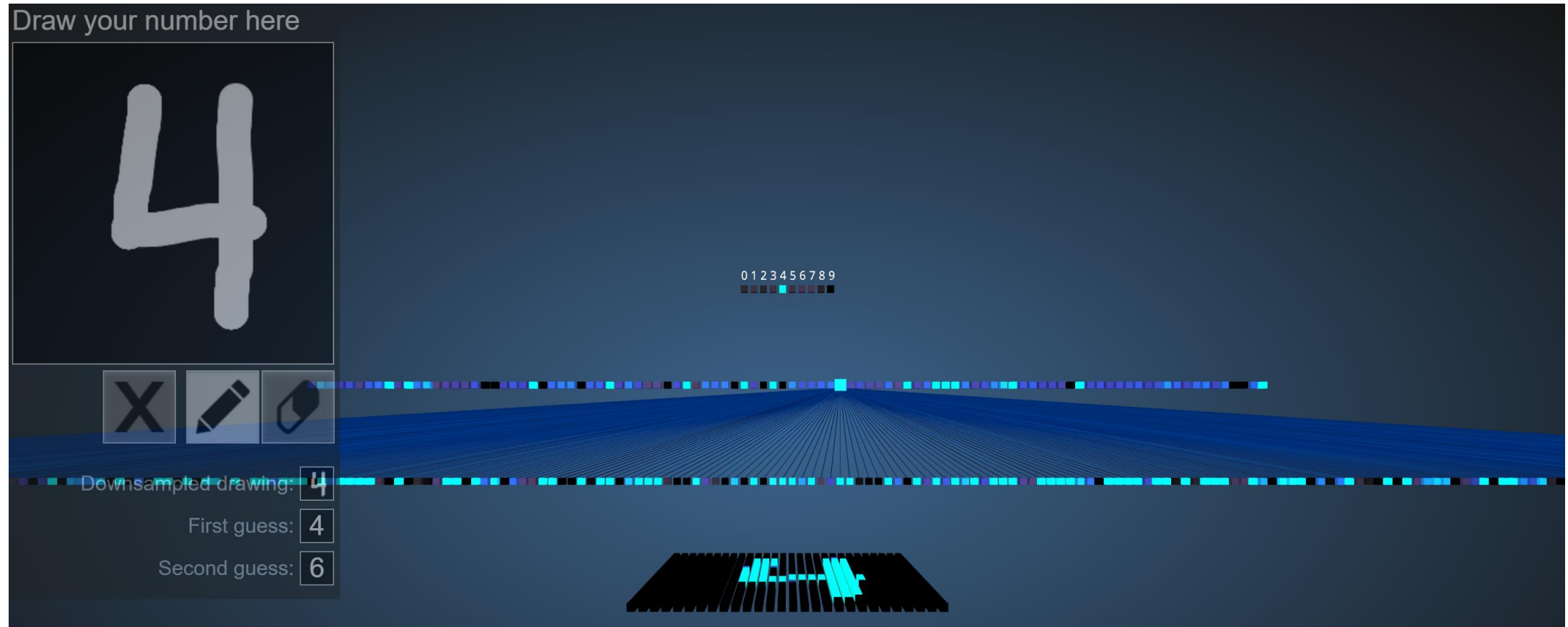


Image Classification

Demo of (Fully-connected) neural network to classify images of hand-written digits

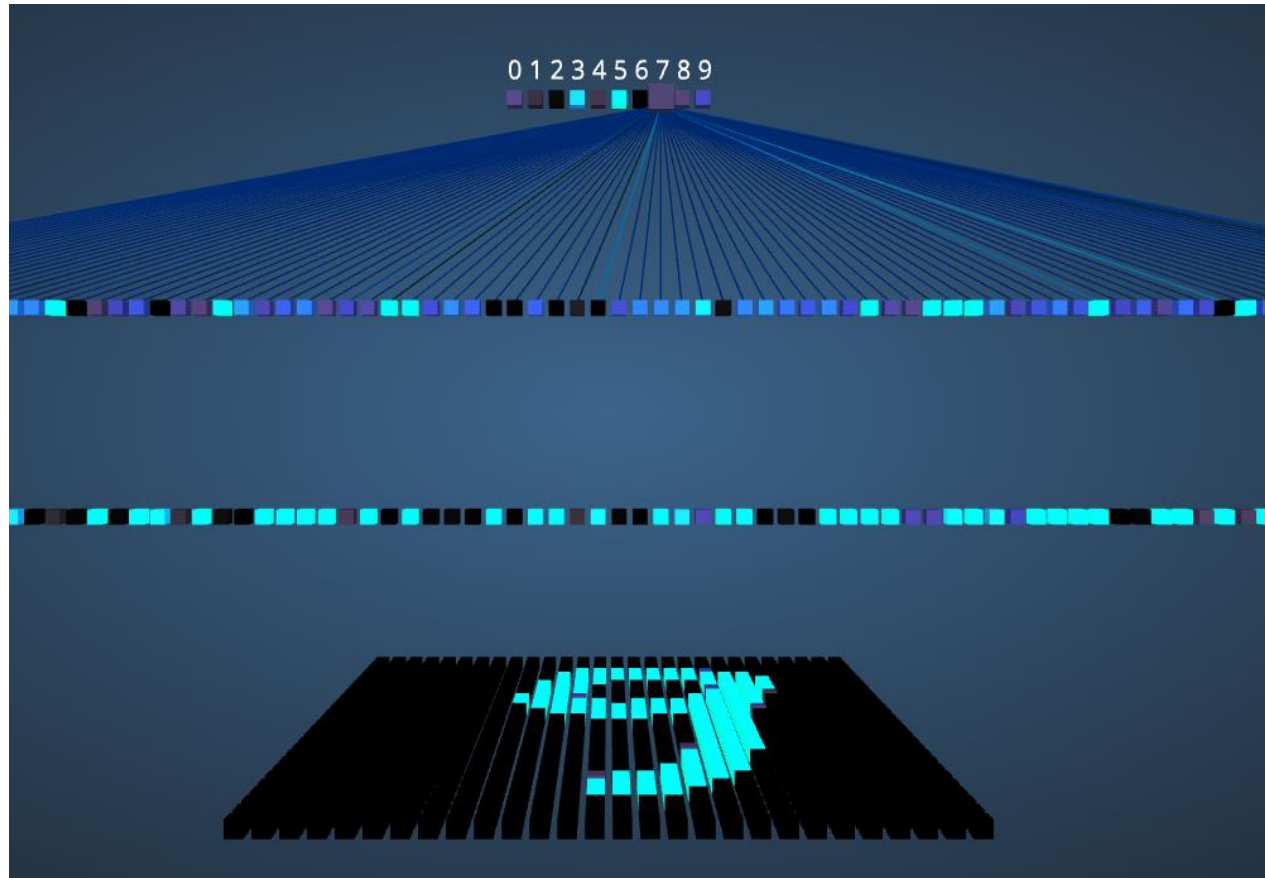
https://adamharley.com/nn_vis/mlp/3d.html



Poll

How many parameters does one neuron in the output layer have?

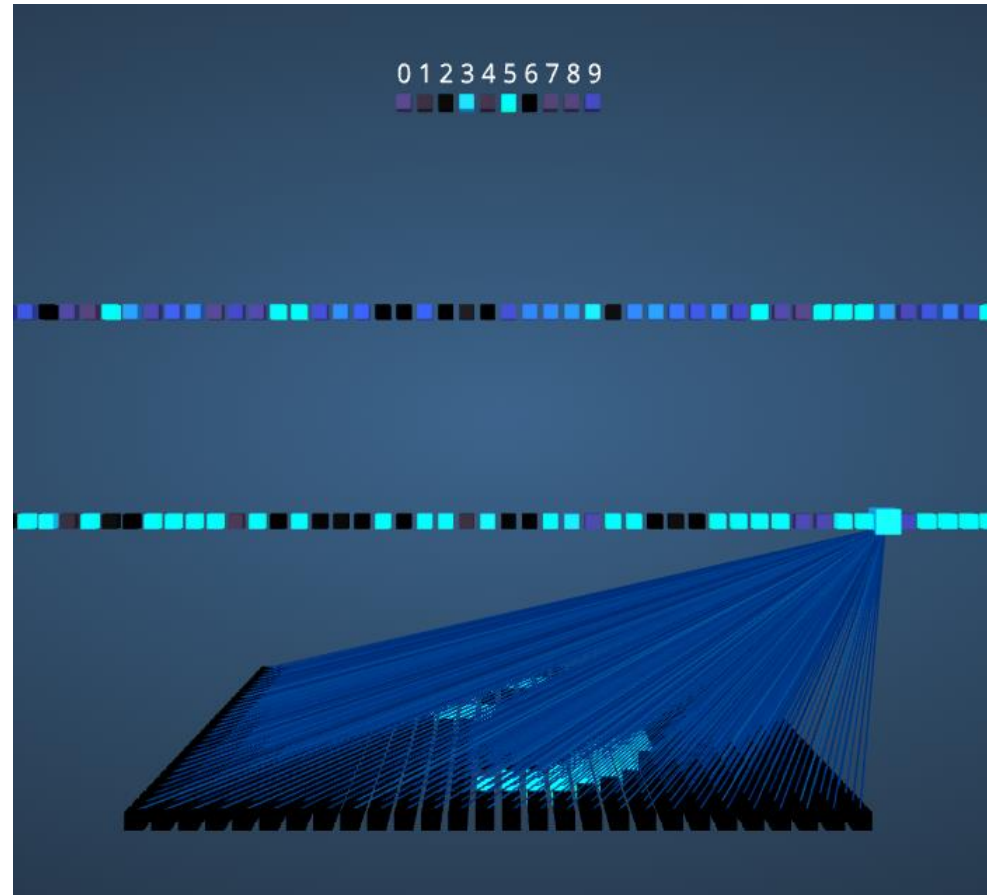
- A. 1
- B. 2
- C. 10
- D. 11
- E. 100
- F. 101



Poll

How many parameters does one neuron in the first hidden layer have?

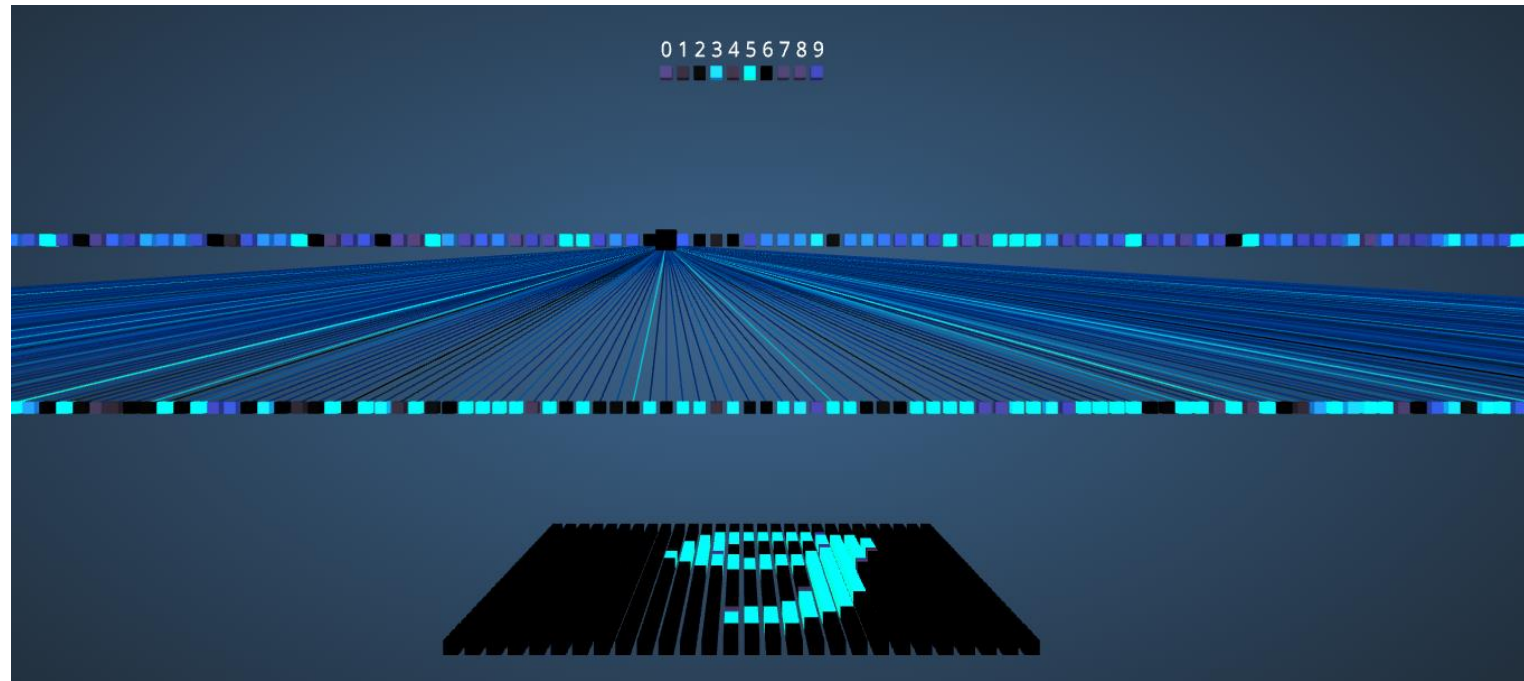
- A. 1
- B. 2
- C. 300
- D. 301
- E. 784
- F. 785



Poll

What do the colors of the lines represent?

- A. Weight
- B. Value from previous neuron

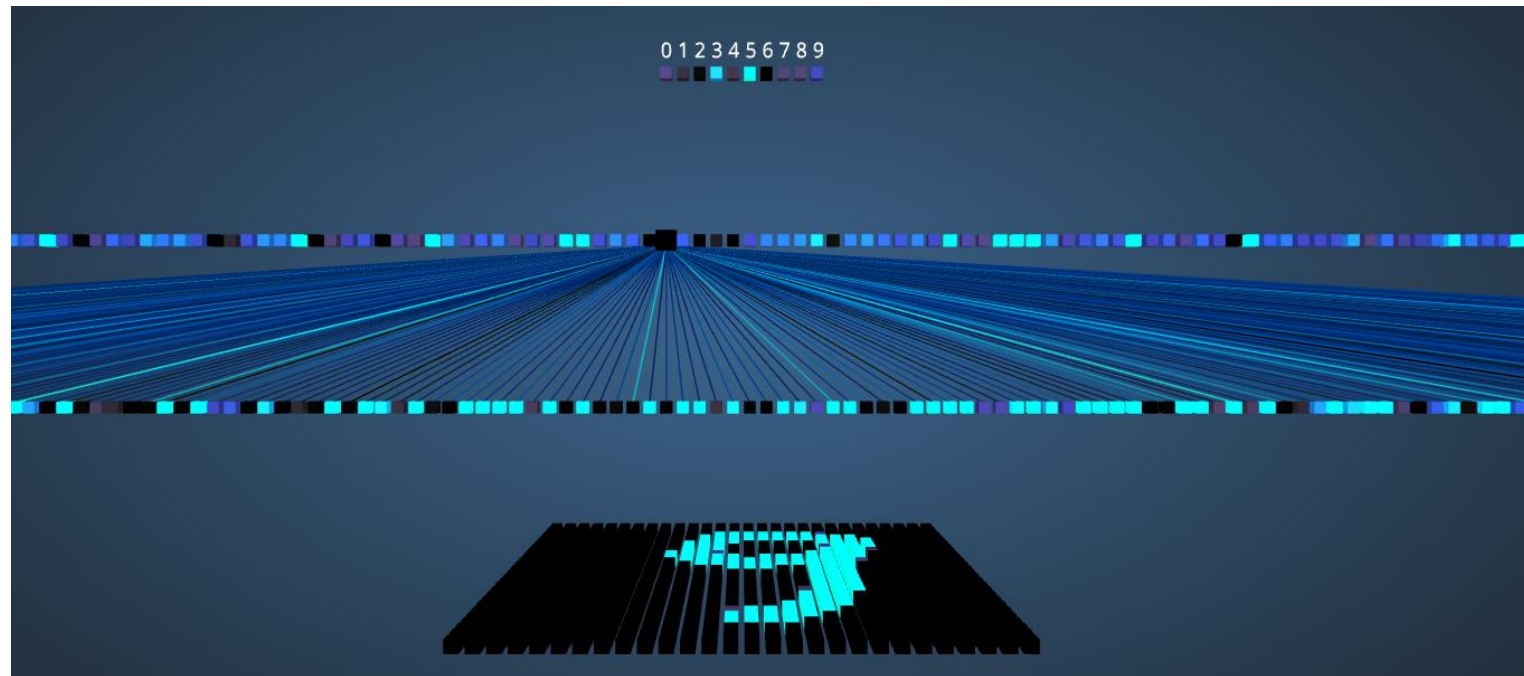


https://adamharley.com/nn_vis/mlp/3d.html

Poll

What do the colors of the neuron squares represent?

- A. Weight
- B. Output value of the neuron
- C. Input value of the neuron



https://adamharley.com/nn_vis/mlp/3d.html