

Problems with ants

The noted gourmet Pangolini Aardvark is preparing a late night snack of “Ant au Chocolat” and “Ant au Fromage”. This requires the use of a five foot pole. One end of the pole is over a bucket of melted chocolate and the other is over a bucket of melted cheese.

Pangolini sprinkles some ants onto the pole. They immediately start scampering along the pole in random directions. If two ants run into each other then they both instantaneously reverse their directions and are now moving away from each other. An ant can change direction many times. Eventually, all of the ants will fall off of one or other end of the pole. If each ant travels at a speed of one inch per second, what is the maximum time until all ants have fallen off?

Suppose now that n ants are placed on a circle of five foot circumference and randomly choose their direction of travel and again reverse direction when they bump into each other. One of the ants is named Alice. What is the probability that Alice is back where she started, one minute after the ants start their scampering.

Back to the pole. Alice starts in the middle of the pole. There are n other ants placed randomly on the pole and they start scampering in random directions. Alice has a cold. When an ant with a cold bumps into another ant, the uninfected ant catches a cold too. What is the expected number of ants who catch cold before they all fall off the pole?

Solutions:

Imagine that each ant carries a distinct flag as it scurries along. When two ants meet they exchange flags. Thus the flags proceed uniformly in one direction along the pole. The ants fall with the flags and so it takes a maximum of one minute for all the flags (and the ants) to fall into the buckets.

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Consider what happens from the point of view of the flags. After 1 minute, the flags must all end up precisely where they started. This means that the set of points on the circle where ants reside is the same after 1 minute as it was at the beginning. However the location of an individual ant is not necessarily the same. We need to figure out what happens to the ants.

Let's give all the ants numbers. Alice is 0. Clockwise from Alice is 1, etc. This ordering does not change over time. Say that initially there are R ants going clockwise and L ants going counterclockwise, so R flags move clockwise and L flags move counterclockwise. Let's consider the aggregate motion of all the ants. In 1 minute, they'll move $5(R - L)$ feet in a clockwise direction. Remember, the set of points containing ants is the same after 1 minute. This proves that

after 1 minute, ant i ends up where ant $(i + R - L) \bmod n$ started. It's the only arrangement preserving the order, the placement, and the necessary distance moved.

So Alice will end up where she started after 1 minute if $R = n$ and $L = 0$ or if $R = 0$ and $L = n$ or if $R = L$. From this we can compute our answer. Let P be the Probability that Alice ends up where she started.

$$P = \begin{cases} 2^{-n} + 2^{-n} + 2^{-n} \binom{n}{n/2} & \text{if } n \text{ is even} \\ 2^{-n} + 2^{-n} & \text{if } n \text{ is odd} \end{cases}$$

There is another way of looking at the problem: Assume w.l.o.g. that Alice moves clockwise initially. Now assume that someone rotates the circle in an anti-clockwise direction at a rate of one revolution per minute. Thus ants moving clockwise now become stationary and the others move at double speed. After one minute Alices original position is where it was, but where is Alice? Suppose that initially there were k ants moving clockwise and $n - k$ ants moving anti-clockwise. After a collision, the ant that was moving anti-clockwise becomes stationary and the other ant moves off. Furthermore, there will be $2k(n - k)$ ant collisions altogether and so Alice will be involved in $2(n - k)$ collisions with other ants during the process. To the observer, there are always k stationary ants at postions X_1, X_2, \dots, X_k on the circle. Alice starts at X_1 and will end up at $X_{[1+2(n-k)] \bmod n}$. So she will end up where she started if either $k = 0$ or $k = n$, giving the same probabilities as before..

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Suppose Alice moves right. The number of ants to right of Alice that catch cold is equal to the number of ants X to the right which start off moving left and if $X \neq 0$ then the number of ants to the left of Alice which catch cold is equal to the number of ants to the left which start off moving right. Thus the number of ants Z which catch cold satisfies

$$\mathbf{E}(Z) = \mathbf{E}(X) + \mathbf{E}(Y)\Pr(X \neq 0) = \frac{n}{4} + \frac{n}{4} \left(1 - \left(\frac{3}{4} \right)^n \right).$$

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Thanks to Mike Schuresko and Chris Peikert for contributing to the solution and to Zhang Hui for pointing out an error. Thanks also to Roger Akers for this amusing thought: "I would assume that the pole is an inverted U. Otherwise, you would have to include gravity into the equation. If the pole is straight and in a vertical attitude, whichever bucket is on top will have its contents flow down the pole sticking the ants to the pole. If the pole is horizontal, the ants may leave the pole anywhere along its length. Also, are you taking into account ants falling off as a product of collisions with one another or a misstep? Interesting problem."