Fair Shares?
A wealthy patron of the arts Phelio has two sons, Aqualo and Lipido. Aqualo likes water colors and Lipido likes oils. Phelio has n water color paintings whose values are $a_1, a_2, \ldots, a_n$ where $a_i \in \{1, 2, \ldots, n\}$ for $i = 1, 2, \ldots, n$. Similarly, Phelio has n oil paintings whose values are $b_1, b_2, \ldots, b_n$ where $b_i \in \{1, 2, \ldots, n\}$ for $i = 1, 2, \ldots, n$.
Phelio has decided to choose two non-empty sets $A, B \subseteq \{1, 2, \ldots, n\}$ and give water color paintings $i \in A$ to Aqualo and oil paintings $i \in B$ to Lipido. There is a constraint. The values of the two sets of paintings are $W = \sum_{i \in A} a_i$ and $O = \sum_{i \in B} b_i$. If $W \neq O$ there will be all hell to pay. Can Phelio always make a gift that avoids such a catastrophe?

Solution:
Phelio can always make a gift that avoids such a catastrophe:
Without loss of generality, let us assume that

$$\sum_{i=1}^{n} a_i \geq \sum_{i=1}^{n} b_i. \tag{1}$$

(Otherwise swap A and B)
Now, for any j it is always possible to express $\sum_{i=1}^{j} b_i$ as $\sum_{i=1}^{j} b_i = \sum_{i=1}^{k} a_i + R_j$ $R_j \in [0, n-1]$ for some $k = k(j) \in [0, n]$. (When $k = 0$ this means that $\sum_{i=1}^{j} b_i \in [0, n-1]$). This is true because all the elements in A and B belong to the set $[1, n]$. Indeed, either $\sum_{i=1}^{j} b_i < a_1$ and then we can directly see that $\sum_{i=1}^{j} b_i = R_j < a_1 \in [1, n]$. Otherwise, take the largest k such that $S_k = \sum_{i=1}^{j} b_j - \sum_{i=1}^{k} a_i \geq 0$. Now $k < n$ by assumption (1). If $S_k \geq n$ then $S_{k+1} \geq 0$, contradiction.

Consider the set of values R_j, for $j \in [1, n]$. If $R_j = 0$, for some j then we are done since $\sum_{i=1}^{j} b_i = \sum_{i=1}^{k(j)} a_i$. If not, there are only $n-1$ possible values for the n quantities $R_1, \ldots, R_n$ and so there exist $j_1 < j_2$ such that $R_{j_1} = R_{j_2}$. But then

$$\sum_{i=1}^{j_1} b_i - \sum_{i=1}^{k(j_1)} a_i = R_{j_1} = R_{j_2} = \sum_{i=1}^{j_2} b_i - \sum_{i=1}^{k(j_2)} a_i$$

and therefore,

$$\sum_{i=j_1}^{j_2} b_i = \sum_{i=k(j_1)}^{k(j_2)} a_i.$$

Hence, it is always possible for Phelio to make a gift to avoid a catastrophe of unfair shares. In fact Phelio can give a contiguous sub-set, which would make Aqualo and Lipido all the more happy.
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