Puzzle 13: Electronic Voting

A storm of controversy has arisen over the use of electronic voting machines in the election for the mayor of Slapville. There are 3 candidates, Moe, Larry and Curley. There are n voters and the winner is a function $f(x_1, x_2, \ldots, x_n)$ of the votes of each voter, $x_j \in \{A, B, C\}$. f is not the simple majority function and the exact function f is known only to the manufacturers of the voting machines.. Professor Pangloss of the Computer Science department of Tickle University claims that the voting system is fair because if $y_i \neq x_i$ for $i = 1, 2, \ldots, n$ then $f(x_1, x_2, \ldots, x_n) \neq f(y_1, y_2, \ldots, y_n)$.

Conspiracy theorists claim that there exists j such that the value of f is completely determined by x_j. Are they right?

Solution: The conspiracy theorists are right. There is something fishy about the election. The problem can be formulated in graph theoretic terms and the proof we give is taken from a paper by Greenwell and Lovász [1].

Given graphs $G_i = (V_i, E_i), i = 1, 2, \ldots, m$ we define their *direct product* $G = G_1 \times G_2 \times \cdots \times G_m$ to be the graph with vertex set $V_1 \times V_2 \times \cdots \times V_m$ and edge set E where there is an edge between $u = (u_1, u_2, \ldots, u_m)$ and $v = (v_1, v_2, \ldots, v_m)$ iff $(u_i, v_i) \in E_i$ for $i = 1, 2, \ldots, m$. We will concentrate on the case where each G_i is a copy of K_r, $r \geq 3$.

Let C denote some finite set. A proper colouring of G assigns a value $f(v) \in C$ to each vertex v so that if $(u, v) \in E$ then $f(u) \neq f(v)$. If $|C| = k$ then we say that f is a k-colouring. In the voting problem, $r = 3$ and the machine defines a 3-colouring of G where $C = \{Moe, Larry, Curlry\}$. Now every propery colouring of G requires at least r colours since we have to colour the vertices $a_i = (i, i, \ldots, i)$, $i = 1, 2, \ldots, r$ differently.

We say that a proper colouring of G is induced by a colouring of some G_j if for vertices u, v, we have $u_j = v_j$ implies $f(u) = f(v)$.

The conspiracy claim is equivalent to that every r-colouring of G is induced by some G_j. This follows from

Theorem 1 *Let H be a connected graph such that any two r-colourings of G differ in at least two vertices. The each r-colouring of $K_r \times H$ is induced by one of the two factors.*

Proof We distinguish two cases:

Case 1 There is an $x \in V(H)$ such that $f(1, x), f(2, x), \ldots, f(r, x)$ are all different. Let y be adjacent to x in H. Then $f(i, y) \neq f(j, x)$ for $i \neq j$ and so $f(i, y) = f(i, x)$ and then the connectivity of H implies that $f(i, z)$ is independent of z.

Case 2 For all $x \in V(H)$ there are $i \neq j$ such that $f(i, x) = f(j, x)$. Fix such an i, j for each x and then denote this colour by $\gamma'(x)$. Fix a vertex $z \in V(H)$ and define

$$\gamma_k(y) = \begin{cases} \gamma'(y) & y \neq z \\ f(k, z) & y = z \end{cases}.$$

Now γ_k is an r-colouring of H. For let u, v be adjacent vertices of H where we can assume that $u \neq z$. Then $\gamma_k(u) = \gamma'(u) = f(i,u) = f(j,u)$ for some $i \neq j$. Moreover, $\gamma_k(v) = f(m,v)$ for some m, no matter if $v = z$ or not. We may assume that $i \neq m$. Then (i,u) and (m,v) are adjacent in the product and so

$$\gamma_k(u) = f(i,u) \neq f(m,v) = \gamma_k(v).$$

The r-colourings $\gamma_1, \gamma_2, \ldots, \gamma_r$ differ only in z; therefore, by our assumption they are identical. Hence $f(i,z)$ is the same for every i. $\square$

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References

[1] D. Greenwell and L. Lovász, Applications of product colouring, *Acta Mathematica Scientiarum Hungaricae* 25 (1974) 355-340.