Rational Creatures

The DNA of a *randimal* is an infinite string of 0's and 1's and is formed as follows: The *randoplast* produces a random string of 100 bits. Then the string doubles itself in length at times 2^{-k} for $k = 1, 2, \ldots,$ seconds. The whole sequence is completed after one second This doubling in length goes as follows:

If the string is currently $x_1, x_2, \ldots, x_m$ then the doubled string is
$x_1, x_2, \cdots, x_m, 1 - x_1, 1 - x_2, \cdots 1 - x_m$.

The behaviour of the creature is rational or irrational, depending on whether the DNA defines a rational or irrational number. Are there any rational randimals?

Solution Unfortunately, all randimals are irrational. We will give two proofs, one is short and self contained. The other relies on properties of the famous *Thue-Morse* or *Prouhet-Thue-Morse* sequence.

Proof 1: Let the sequence after r iterations be denoted by $\mathbf{x}^{(r)}$. We claim that no matter what the initial sequence $\mathbf{x}^{(0)} = (x_1 x_2 \cdots x_k)$, the final sequence $\mathbf{x}^{(\infty)}$ defines an irrational number. Suppose to the contrary, that there exists an initial sequence for which the final sequence $\mathbf{x}^{(\infty)}$ can be expressed as $uvvvvvvvvvvv\cdots$ for some finite sequences u, v. Let $|v| = 2^a b$ where $a \geq 0$ and b is odd.

We show first that a must be strictly positive. Indeed, suppose that $a = 0$ and $b = 2p+1$ and w.l.o.g. that v has more 1's than 0's. Now after q iterations of our procedure, the string $\mathbf{x}^{(q)}$ produced so far will be of length $\ell = 2^q k$ and will have an equal number of 0's and 1's. Suppose that $\ell \in [|u| + t|v| + 1, |u| + (t + 1)|v|]$ where $t = \lfloor (\ell - |u|)/|v| \rfloor$. Then $\mathbf{x}^{(\ell)}$ has at least $(p + 1)t$ 1's and at most $pt + |u| + |v|$ 0's. If q is large enough so that $t > |u| + |v|$ then $\mathbf{x}^{(q)}$ will have more 1's than 0's, contradiction.

We can assume that $|u|$ is even, otherwise replace $\mathbf{x}^{(0)}$ by $\mathbf{x}^{(1)}$ as the initial sequence. Now assume that our initial sequence has been chosen so that a is as small as possible. If we delete the even entries of our initial sequence, and run our process, then we will produce a sequence $u'v'v'v'\cdots$ where u', v' are obtained from u, v by deleting entries in even positions. This reduces a to $a-1$, contradicting our assumption that a is as small as possible.

Proof 2: This uses a property of the famous *Thue-Morse* or *Prouhet-Thue-Morse* sequence. This is the sequence $\mathcal{S}$ of 0's and 1's that you get if you start the above process with a $\mathbf{x}^{(0)} = 0$ i.e.

0 1 10 1001 10010110 1001011001101001...

It is known that $\mathcal{S}$ is *cube free* i.e. $\mathcal{S}$ does not contain a consecutive substring of the form www, for a substring w.

If $|\mathbf{x}^{(0)}| = m$ then the subsequence $\mathcal{X} = (x_{mn+1}, n \geq 0)$ is either $\mathcal{S}$ or $\mathbf{1} - \mathcal{S}$ and as such is cube free. This implies that $\mathbf{x}^{(\infty)}$ is irrational. Suppose for example that $\mathbf{x}^{(\infty)} = xyyyyyy\cdots$ for some strings x, y. We can assume that $|x|$ is a multiple of m. Indeed we can assume $|x|, |y| \geq m$ by adding y's to x and then replacing y by multiples of y if necesary. Then, if $|x| = mp + q$ then we let $x' = xz$ where z is the first $m - q$ bits of y and then let $y' = wz$ where w is the last $|y| - (m - q)$ bits of y. This yields $\mathbf{x}^{(\infty)} = x'y'y'y'\cdots$ and because each y'

starts with a member of $\mathcal{X}$ and contains the same number of elements we see immediately that this implies $\mathcal{S}$ is not cube free, contradiction.

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