LEVEL SPACINGS FOR CAYLEY GRAPHS

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Abstract. We investigate the eigenvalue spacing distributions for randomly generated 4-regular Cayley graphs on $SL_2(\mathbb{F}_p)$, S_{10}, and large cyclic groups by numerically calculating their spectra. We present strong evidence that the distributions are Poisson and hence do not follow the Gaussian orthogonal ensemble. Among the Cayley graphs of $SL_2(\mathbb{F}_p)$ that we consider are the new expander graphs recently discovered by Y. Shalom. In addition, we use a Markov chain method to generate random 4-regular graphs, and observe that the average eigenvalue spacings are closely approximated by the Wigner surmise.

Key words. Random matrices, Cayley graphs, expander graphs, spacing distribution, Gaussian ensemble, Wigner surmise.

AMS(MOS) subject classifications. Primary 05C25, 20C40, 68R10; Secondary 20B25, 20D06, 20C30.

1. Introduction. One of the most remarkable numerical discoveries of the recent past is Odlyzko's finding that the spacings of the zeros of the Riemann zeta function closely follow the Gaussian unitary ensemble of random matrix theory [16]. As a result of this work, attention has turned to the spacing distributions for the spectrum of other natural classes of operators, in the hope of making similar connections with other number theoretic objects.

One direction of related work is towards the analysis of the eigenvalue spacing distribution for the Laplacian on different manifolds; see [18] for a survey of many recent results of this type and an extensive bibliography. The motivation for our work comes from the particular case of interest of the Laplacian on $SL_2(\mathbb{Z})\backslash\mathbb{H}$, where $\mathbb{H}$ denotes the hyperbolic upper half plane. This is the main example of an *arithmetic surface*, which is a hyperbolic surface given as the quotient of the upper half plane by an arithmetic subgroup of $SL_2(\mathbb{R})$.

In general, the statistical behavior of eigenvalue spacings for natural families of operators falls into two main classes, Poisson and the Gaussian ensemble. The Gaussian orthogonal ensemble (GOE) governs random symmetric matrices while the Gaussian unitary ensemble (GUE) applies to random complex Hermitian matrices. The density of the spacings in the Poisson case, where the spacings are normalized to have unit mean, is exponential, e^{-x}, and for the Gaussian orthogonal ensemble the density

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is well-approximated by the *Wigner surmise* $\frac{\pi s}{2}e^{-\pi x^2/4}$ [14]. More generally, Katz and Sarnak [8] have recently investigated the eigenvalue spacings for the classical groups, as well as connections to zeta functions for curves over finite fields. Diaconis and Shahshahani [3] have analyzed the eigenvalue distribution for a randomly chosen matrix from classical groups. The eigenvalues have a physical interpretation as energy levels, and Poisson behavior of the spacings is usually thought of as characteristic of integrable systems, while GOE corresponds to chaotic systems (cf. [1,4,14] and the many references therein).

Computations of Schmit [20] indicate that the spacing distribution for $SL_2(\mathbb{Z})\backslash\mathbb{H}$ should be Poisson. Along these lines, results of Luo and Sarnak [13] and Rudnick and Sarnak [17] indicate that this is the case for any arithmetic surface. The exact behavior of the spacings is still an open question.

In this paper we investigate the eigenvalue spacings for the adjacency matrices of 4-regular (2-generator) Cayley graphs for three families of groups: $SL_2(\mathbb{F}_p)$, symmetric groups and cyclic groups. In all three cases our computations indicate that generically (i.e., for randomly chosen pairs of generators), the spacing distribution for the associated adjacency matrices is in close agreement with Poisson behavior.

Our primary and most detailed examples come from Cayley graphs on $SL_2(\mathbb{F}_p)$, naturally thought of as discrete approximations to the spectral behavior in the continuous setting of $SL_2(\mathbb{Z})\backslash\mathbb{H}$. This sort of analogy is suggested by the machinery developed in the successful application of Selberg's Theorem [22] to the discovery of expander graphs built as Cayley graphs of $SL_2(\mathbb{F}_p)$. Here the main tool is the transfer of the lower bound on the first positive eigenvalue of the Laplacian to a uniform bound on the first nonzero eigenvalue for a family of graphs obtained as quotients of Cayley graphs on $SL_2(\mathbb{Z})$. See [11] for an excellent treatment of this construction and a thorough bibliography. Our computations treat generator sets to which Selberg's Theorem applies, as well as some sets to which it does not apply. This includes an investigation of the recent expander constructions due to Shalom [24].

For comparison we have also computed the spectra for randomly generated 4-regular graphs, using Markov chain methods to generate the graphs. These all closely follow the GOE, and this is in agreement with the extensive computations performed by Jakobson, Miller, Rivin and Rudnick [5]. Our computations on $SL_2(\mathbb{F}_p)$ and the symmetric groups are made possible by the use of the representation theory for these groups (see Section 2). In brief, we compute the spectrum as the union of spectra of individual Fourier transforms of the characteristic function of the generating set. In addition, we investigate the spectra of some of these individual transforms for $SL_2(\mathbb{F}_p)$. This includes the spectrum of the expander graphs built as the action of $SL_2(\mathbb{F}_p)$ on the projective line. In all cases the distributions appear to exhibit Poisson behavior.

After outlining our use of Fourier analysis in Section 2, we present our computations in Section 3. We summarize our results in the form of conjectures in Section 4.

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2. Cayley graphs and Fourier analysis. As in [9,10], to analyze the spectrum of Cayley graphs, we exploit the fact that the adjacency matrix can be viewed as the Fourier transform of the delta function on the generators at the right regular representation. Any representation is equivalent to a direct sum of irreducible representations. Thus, if we are able to compute Fourier transforms at any irreducible representation for the defining group G, then we can recover the complete spectrum by only computing the spectrum of each individual Fourier transform.

2.1. The general case. Let G be a finite group and let $\mathcal{S} \subset G$ generate G. The *Cayley graph* $X = X(G, \mathcal{S})$ *for* G *with respect to* $\mathcal{S}$ is the undirected graph with vertex set equal to G, such that there is an edge between a and b in X if and only if $as = b$ for some $s \in \mathcal{S} \cup \mathcal{S}^{-1}$. Equivalently, the adjacency matrix of $X(G, \mathcal{S})$ has a one in the (a, b) entry if and only if $as = b$ for some $s \in \mathcal{S} \cup \mathcal{S}^{-1}$. Let ρ_{reg} denote the right regular representation of G, computed with respect to the basis of delta functions on G. Then it is not difficult to see we have the following expression for the adjacency matrix of $X(G, \mathcal{S})$, denoted $\Gamma(G, \mathcal{S})$:

$$\Gamma(G, \mathcal{S}) = \sum_{s \in \mathcal{S} \cup \mathcal{S}^{-1}} \rho_{reg}(s). \tag{2.1}$$

The righthand side of (2.1) is also the *Fourier transform* of the characteristic function for $\mathcal{S} \cup \mathcal{S}^{-1}$.

Direct computation of the spectrum of the $|G| \times |G|$ matrix $\Gamma(G, \mathcal{S})$ requires $O(|G|^3)$ operations (cf. [26]). For example, for $SL_2(\mathbb{F}_p)$, since $|SL_2(\mathbb{F}_p)| = O(p^3)$, this means $O(p^9)$ operations. This cost quickly becomes prohibitive as p gets large. However, by using the tools of representation theory, we may instead compute the elements of an equivalent block diagonal matrix and realize the entire spectrum as the union of the spectra of a subset of the blocks.

Representation theory gives a simultaneous block diagonalization of the matrices $\rho_{reg}(s)$ as

$$\rho_{reg}(s) \sim \begin{pmatrix} B_1(s) & 0 & \cdots & 0 \\ 0 & B_2(s) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_h(s) \end{pmatrix} \tag{2.2}$$

with

$$B_i(s) = \begin{pmatrix} \rho_i(s) & 0 & \cdots & 0 \\ 0 & \rho_i(s) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \rho_i(s) \end{pmatrix} \tag{2.3}$$

where $\{\rho_1, \ldots, \rho_h\}$ is a complete set of irreducible matrix representations of G and the matrix $B_i(s)$ in (2.3) has $\deg(\rho_i)$ copies of $\rho_i(s)$ on the diagonal. Consequently,

$$\mathit{spectrum}(X(G, \mathcal{S})) = \bigcup_{i=1}^{h} \mathit{spectrum}\left(\sum_{s \in \mathcal{S} \cup \mathcal{S}^{-1}} \rho_i(s) \right). \tag{2.4}$$

The degree of the largest irreducible representation of G is bounded above by $|G|^{1/2}$ (cf. [23]). Thus, if we are able to directly compute the matrices $\rho_i(s)$, we are able to reduce the computation from $O(|G|^3)$ operations to a more manageable $O\big(|G| \cdot \max_i \deg(\rho_i)\big) \leq O(|G|^{3/2})$ operations.

Remark. In the case in which $\mathcal{S} \cup \mathcal{S}^{-1}$ is a union of conjugacy classes of G, the adjacency matrix can be completely diagonalized and the eigenvalues can be computed as certain character sums over G (see e.g., [2]).

The above discussion is completely general and indicates the methodology used in each of our three Cayley graph examples, built on the groups S_n, $\mathbb{Z}/n\mathbb{Z}$ and $SL_2(\mathbb{F}_p)$. The explicit representation theory of the first two groups is fairly well known and will not be reviewed here. The representation theory of the cyclic groups can be found in any elementary representation theory text (see e.g. [23]) and that of the symmetric groups has been given extensive treatment as well (see e.g. [6]). On the other hand, the representation theory of $SL_2(\mathbb{F}_p)$ is not nearly as well known, and the following discussion requires that we explain it in some detail.

2.2. Representation theory for $SL_2(\mathbb{F}_p)$. For $G = SL_2(\mathbb{F}_p)$, the irreducible representations of $SL_2(\mathbb{F}_p)$ occur in two families, the *discrete series* and *principal series.* The distinction depends upon the restriction of an irreducible representation to the Borel subgroup $B < SL_2(\mathbb{F}_p)$ of upper triangular matrices.

An irreducible representation of $SL_2(\mathbb{F}_p)$ is said to be from the principal series if its restriction to B contains the trivial representation. Otherwise, it is said to be from the discrete series. The principal series representations occur as components of induced 1-dimensional representations from B, all but two of which are irreducible. This gives the trivial representation, one representation of degree p, two representations of degree $(p+1)/2$, and $(p-3)/2$ representations of degree $p+1$.

The discrete series is less easily explained, but suffice it to say that the representations are in close correspondence with the characters of the non-split torus in $SL_2(\mathbb{F}_p)$ (cf. [15], Ch. 2, Section 5). There are two such representations of degree $(p-1)/2$ and $(p-1)/2$ representations of degree $p-1$. Explicit representations are needed to apply (2.2) and (2.3). These can be found in [15] and [25] and are the basis of our implementation (cf. [9]).

Knowledge of the representations of $SL_2(\mathbb{F}_p)$ gives the irreducible representations of $PSL_2(\mathbb{F}_p)$. More precisely, if ρ is an irreducible matrix representation of $SL_2(\mathbb{F}_p)$ and $-I$ is in the kernel of ρ, where $I = \left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)$, then ρ is constant on cosets $SL_2(\mathbb{F}_p)/\{\pm I\}$ and as such gives an irreducible representation of $PSL_2(\mathbb{F}_p)$. Under this identification, the set $\{\rho \mid \{\pm I\} \subset \ker(\rho)\}$ gives a complete set of inequivalent irreducible representations of $PSL_2(\mathbb{F}_p)$. We use this correspondence in Section 3 when calculating the spectra for the Ramanujan graphs constructed by Lubotzky, Phillips and Sarnak.

3. Numerical evidence. The main results of this paper are computations of the spacings for various Cayley graphs, made possible by the techniques outlined in Section 2. As we stated in the introduction, our computations are of two types:

(1) The computation of spacing distributions for particular 2-generator Cayley graphs for $\mathbb{Z}/n\mathbb{Z}$, S_{10}, and $SL_2(\mathbb{F}_p)$.
(2) The computation of spacing distributions for particular Fourier transforms for 2-generator Cayley graphs on $SL_2(\mathbb{F}_p)$.

For a generating subset $\mathcal{S} \subset G$ we denote the eigenvalues (without multiplicities) of $\Gamma(G, \mathcal{S})$ as $\lambda_0 > \lambda_1 > \cdots > \lambda_N$, and let $P(S)$ denote the "empirical" cumulative distribution function (cdf) for the eigenvalue spacings, so that

$$P(S) = \frac{1}{N}\sum_{j=1}^{N}[\lambda_{j-1} - \lambda_j \leq S] \tag{3.1}$$

where $[a \leq b]$ is one if $a \leq b$ and is zero otherwise. We assume the eigenvalues are normalized so that the spacings have mean one:

$$\frac{1}{N}\sum_{j=1}^{N}(\lambda_{j-1} - \lambda_j) = 1\,. \tag{3.2}$$

For the group $SL_2(\mathbb{F}_p)$ we distinguish among various kinds of generators. *Global* or *Selberg* generators are generators for families of Cayley graphs obtained as the projection of a single Cayley graph on $SL_2(\mathbb{Z})$; *non-Selberg* generators are an infinite family of generators for $SL_2(\mathbb{F}_p)$ as $p \to \infty$ that are not the projection of a single set of generators for $SL_2(\mathbb{Z})$; *random* generators are pairs of generators for $SL_2(\mathbb{F}_p)$ generated by a simple randomized algorithm. In general, random generators are non-Selberg.

Besides the computations of the various $P(S)$, we also include some new data on a particularly interesting non-Selberg generating pair recently discovered by Y. Shalom [24]. These turn out to be (in terms of the numerical analysis of the second-largest eigenvalue) among the best expanders built as Cayley graphs of $SL_2(\mathbb{F}_p)$ discovered to date.

We compare and contrast the behavior of our families of 4-regular Cayley graphs with that of random 4-regular graphs. The former almost always show strong agreement with Poisson behavior, while for the latter, the spacing distributions show GOE behavior (see also [5]).

3.1. Random graphs. To generate random 4-regular graphs (not Cayley graphs) we used a Markov chain method. The states of the Markov chain are the (labeled) k-regular graphs, and two graphs are connected by a single step of the random walk if and only if the symmetric difference of their edges is a cycle of length 4. The random walk can be described in terms of the incidence matrices of the graphs. Recall that the incidence matrix of a graph $\Gamma(E, V)$ is the $|V| \times |E|$ matrix $I(\Gamma)$ where the column corresponding to edge (i, j) has a 1 in the i^{th} and j^{th} row, and 0's elsewhere. If I is the state of the random walk, two rows $1 \leq i < j \leq |V|$ and two columns $1 \leq k < l \leq |E|$ are chosen uniformly at random. If $I_{ik} = I_{jl} = 1$ and $I_{il} = I_{jk} = 0$ then the chain moves to the state with $I_{ik} = I_{jl} = 0$ and $I_{il} = I_{jk} = 1$ unless a double edge is formed by doing so. Similarly, if $I_{ik} = I_{jl} = 0$ and $I_{il} = I_{jk} = 1$ then the chain moves to the state with $I_{ik} = I_{jl} = 1$ and $I_{il} = I_{jk} = 0$, again unless a double edge would be formed by this move. In all other cases the walk remains in the same state.

It is proved in [7] that this random walk is rapidly mixing, using the technique of canonical paths to estimate conductance. While this results in an algorithm which is polynomial in the size of the graphs, the exponent is too large to enable this result to yield a stopping criterion for the graphs we generate. As a matter of practicality, we simply run the chain for a large number (10^8) of steps to generate each graph. After stopping the chain, we test to make sure the graph is connected (it is with high probability).

Since the Cayley graphs for $SL_2(\mathbb{F}_{157})$ yield on the order of 25,000 eigenvalues and 19,000 intervals, we generated 10 random 4-regular graphs on 2,000 vertices and averaged the intervals to obtain comparable statistics. The resulting cumulative distribution function is shown in Figure 1, where it is compared with the Wigner surmise.

3.2. Random graphs for $SL_2(\mathbb{F}_p)$. To choose generating pairs of $SL_2(\mathbb{F}_p)$ uniformly at random, we use the algorithm described in [9]. This algorithm first selects two group elements $a, b \in SL_2(\mathbb{F}_p)$ uniformly at random using the Bruhat decomposition of $SL_2(\mathbb{F}_p)$, and then checks whether $\{a, b\}$ generates the group by verifying that $\{\pi(a), \pi(b)\}$ does not generate one of the six possible subgroups of $PSL_2(\mathbb{F}_p)$, where $\pi : SL_2(\mathbb{F}_p) \to PSL_2(\mathbb{F}_p)$ is the natural projection. We refer to [9] for details.

We generated random Cayley graphs using this algorithm for several primes $p \geq 150$, and observed that all of the graphs closely followed the exponential distribution. In Figure 2 we show the cumulative distribution

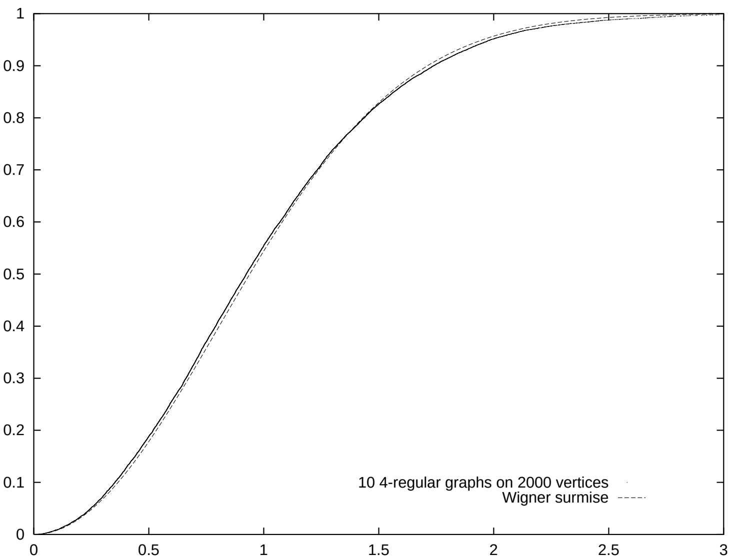

FIG. 1. *The cumulative distribution function $P(S)$ for the eigenvalue spacings of the 10 4-regular graphs on 2,000 vertices, generated by running the Markov chain for* 10^8 *steps. The average value of λ_1 was 0.863389. The dashed line is the cdf for the Wigner surmise,* $1 - \exp(-\frac{\pi S^2}{4})$.

function for a typical example.

Remark. In Figure 2, as in all of the cumulative distributions that we present, we have omitted the spectral gap $k - \lambda_1$ from the calculations. Asymptotically, as $p \to \infty$, this gap does not contribute to the cdf, although it changes the mean of the distribution.

3.3. Explicit generators for $SL_2(\mathbb{F}_p)$. When we computed the spacing distributions for both Selberg and non-Selberg generators, and for the generators recently discovered by Shalom [24], we found that they followed the Poisson behavior very closely. The generators for $SL_2(\mathbb{F}_p)$ we used were the following:

$$\left\{ \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \right\} \qquad \textit{Selberg} \tag{3.3}$$

$$\left\{ \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \right\} \qquad \textit{non-Selberg} \tag{3.4}$$

$$\left\{ \begin{pmatrix} 1+\omega & -1 \\ -\omega & 1 \end{pmatrix}, \begin{pmatrix} -2\omega & -\omega \\ 1+\omega & 1+\omega \end{pmatrix} \right\} \qquad \textit{Shalom} \tag{3.5}$$

where ω is a primitive cube root of unity $(mod\ p)$. For $p = 199$, the distribution for Shalom's generators is shown in Figure 3. The curves for the other generators are very similar.

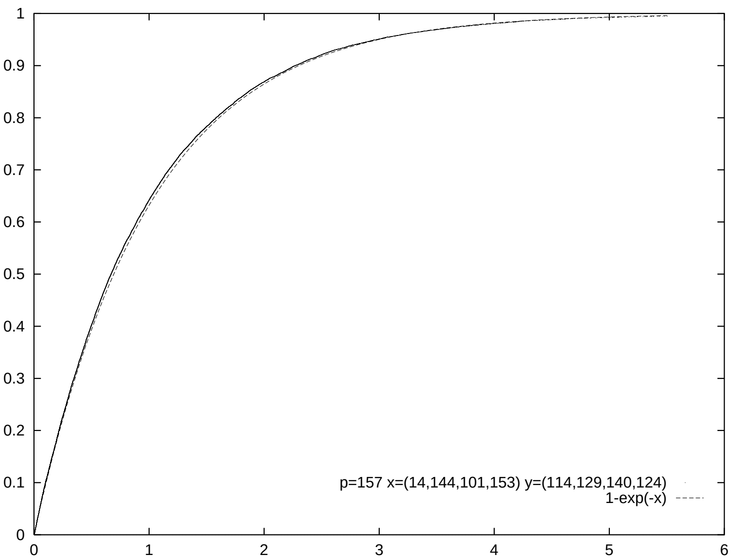

FIG. 2. *The cumulative distribution function $P(S)$ for the eigenvalue spacings of the single, randomly chosen generating pair* $a = \begin{pmatrix} 14 & 144 \\ 101 & 153 \end{pmatrix}$, $b = \begin{pmatrix} 114 & 129 \\ 140 & 124 \end{pmatrix}$ *for* $SL_2(\mathbb{F}_{157})$. *The curve does not include the spacing between the first and second eigenvalues, which in this case was* $1 - 0.879090$. *The dashed line is the curve* $1 - e^{-S}$.

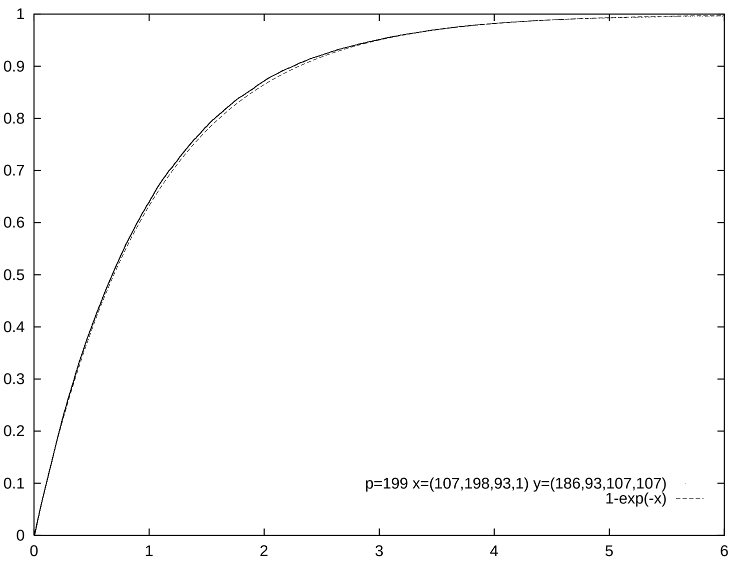

FIG. 3. *The cumulative distribution function $P(S)$ for generators (3.5) with* $p = 199$, *so* $a = \begin{pmatrix} 107 & 198 \\ 93 & 1 \end{pmatrix}$, $b = \begin{pmatrix} 186 & 93 \\ 107 & 107 \end{pmatrix}$. *The second-largest eigenvalue is* $\lambda_1 \approx 0.886048$.

p	λ_1	p	λ_1
13	0.921721	109	0.908280
19	0.864292	127	0.886755
31	0.879056	139	0.887515
37	0.894603	151	0.880739
43	0.894659	157	0.888702
61	0.882493	163	0.881105
67	0.890386	181	0.890412
73	0.903236	193	0.887729
79	0.881429	199	0.886048
97	0.899836	211	0.881852
103	0.883868	223	0.882423

TABLE 1
λ_1 *for a sequence of Shalom's expanders.*

The second-largest eigenvalue λ_1 is shown for a few of the small primes $p \equiv 1 \ (mod\ 4)$ in Table 1. These numbers indicate that, except for the known Ramanujan graphs, these generators are perhaps the best explicit 4-regular graphs for $SL_2(\mathbb{F}_p)$ that have been obtained. For $PSL_2(\mathbb{F}_p)$, the explicit Ramanujan graphs of Lubotzky, Phillips and Sarnak have better separation. For comparison, the second-largest eigenvalue for the LPS generators

$$\frac{1}{\sqrt{3}}\begin{pmatrix} i & 1 \pm i \\ -1 \pm i & -i \end{pmatrix}, \quad \frac{1}{\sqrt{3}}\begin{pmatrix} i & -1 \pm i \\ 1 \pm i & -i \end{pmatrix} \tag{3.6}$$

where $i = \sqrt{-1}$, are shown for several values of $p \equiv 3 \ (mod\ 4)$ in Table 2.

Not all of the spacing distributions that we observed for $SL_2(\mathbb{F}_p)$ were so closely Poisson. As an example, the plot in Figure 4 shows the spacing distribution for the Selberg type generators

$$\left\{ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \right\} \tag{3.7}$$

for $p = 199$. The second-largest eigenvalue for this graph is $\lambda_1 \approx 0.977554$. Note that graphs for this pair of generators are poor expanders, and thus are not typical of the spectral behavior of random Cayley graphs for $SL_2(\mathbb{F}_p)$.

3.4. Individual transforms for $SL_2(\mathbb{F}_p)$. The data presented above strongly indicate that the average spacing associated with the Fourier transforms of the delta function supported on the generating set is asymptotically Poisson. In this section we investigate the behavior of the individual transforms, and present sample calculations for the LPS graphs $X^{p,q}$

p	λ_1
13	0.832880
37	0.863086
61	0.865375
73	0.862093
97	0.864023
109	0.863180
157	0.861790
181	0.863598
193	0.862532
229	0.865491
241	0.864479

TABLE 2
λ_1 for 4-regular Ramanujan graphs (3.6) on $PSL_2(\mathbb{F}_p)$. The Ramanujan bound is $\sqrt{3}/2 \approx 0.8660254$.

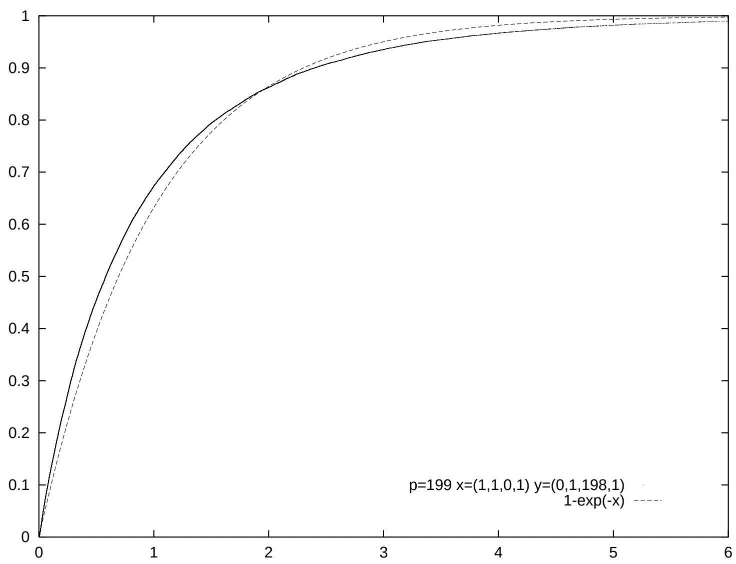

FIG. 4. *$P(S)$ with generators $a=\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, b = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$ for $SL_2(\mathbb{F}_{199})$. The second-largest eigenvalue is $\lambda_1 \approx 0.977554$.*

[12,19]. Of particular interest is the transform at the principal series representation induced from the identity, since this is associated with a graph on the projective line.

To explain, we recall that $SL_2(\mathbb{F}_q)$ acts on the projective line $\mathbb{P}^1(\mathbb{F}_q) = \{0, 1, \ldots, q-1, \infty\}$ by fractional linear transformations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \omega = \frac{a\,\omega + b}{c\,\omega + d}.$$

Let B denote the Borel subgroup $\left\{ \left(\begin{smallmatrix} \alpha & b \\ 0 & \alpha^{-1} \end{smallmatrix} \right) \right\}$ and let the matrices

$$s_u = \begin{pmatrix} 0 & 1 \\ -1 & -u \end{pmatrix}, \; u = 0, 1, \ldots, q-1 \text{ and } s_\infty = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

be a fixed set of coset representatives of $B \backslash SL_2(\mathbb{F}_q)$, which we identify with $\mathbb{P}^1(\mathbb{F}_q)$. In [9] we constructed the principal series representations of $SL_2(\mathbb{F}_q)$ by inducing characters from B, and expressing these matrices in terms of their action on $\{s_u\}$. Very briefly, if $\psi : \mathbb{F}_q^\times \to \mathbb{C}$ is a character of B, then the induced representation $\rho_\psi = \psi \uparrow B$ is given as

$$\rho_\psi(g) s_u = \psi(f(u, g)) s_{g \cdot s_u} \tag{3.8}$$

for some function $f(u, s) \in \mathbb{F}_q^\times$. In particular, $\widehat{\delta_{\mathcal{S}}}(\rho_1)$ is the adjacency matrix $\Gamma(\mathbb{P}^1(\mathbb{F}_q), \mathcal{S})$; we refer to [9] for details. If $\Gamma(PSL_2(\mathbb{F}_q), \mathcal{S})$ is a good expander, then $\Gamma(\mathbb{P}^1(\mathbb{F}_q), \mathcal{S})$ is also a good expander since its spectral gap is at least as large. In particular, the explicit Ramanujan graphs for $PSL_2(\mathbb{F}_q)$ restrict to give Ramanujan graphs on $\mathbb{P}^1(\mathbb{F}_q)$ [19].

Our experience [9,10] has been that the spectral properties of the principal and discrete series representations are almost identical, and that the spectral properties of any individual Fourier transform are representative of those of the full Fourier transform. Our results here are consistent with these previous findings. Figure 5 shows the spacing distribution for the 38-regular LPS graphs $X^{37,1009}$ for $PSL_2(\mathbb{F}_{1009})$ at two principal series and two discrete series representations.

3.5. Spacings for the symmetric and cyclic groups. In addition to our calculations for $SL_2(\mathbb{F}_p)$, summarized in the previous sections, we have computed the spacings for random Cayley graphs on the symmetric group $\mathcal{S}_n$ and the cyclic group $\mathbb{Z}/n\mathbb{Z}$ and have found that the distributions are again Poisson.

In the case of the symmetric group, we computed the spectra of 4-regular Cayley graphs on $\mathcal{S}_{10}$ with respect to the following generating pairs (randomly chosen using the GAP programming language [21]):

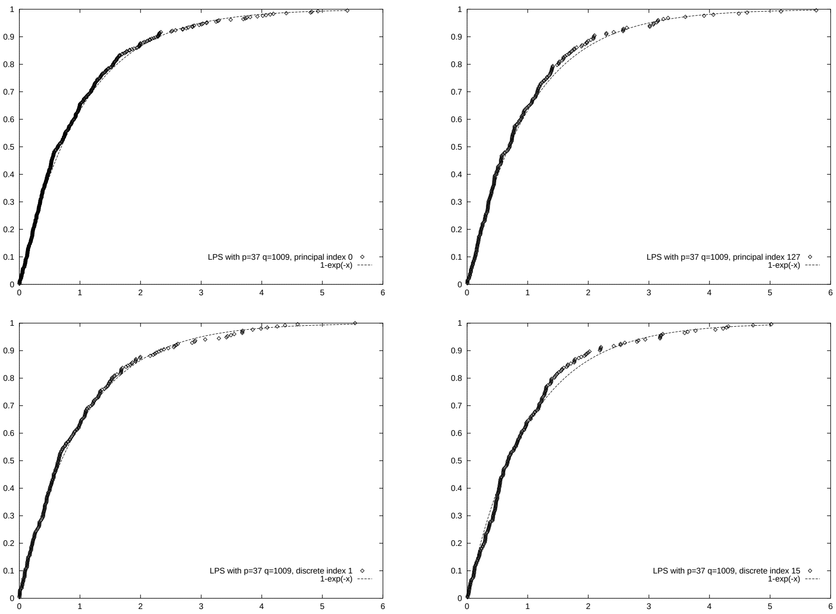

FIG. 5. *$P(S)$ for individual Fourier transforms, for LPS generators with $p = 37$ and $q = 1009$. Top row: two principal series representations, action on projective line on the left. Bottom row: two discrete series representations.*

(1 10 9 2 8 4 6 3 7)	(1 2 6 10 8 5 7 4)
(1 8 5 3 6 9 2 10 7 4)	(1 4 9 5 8 10 2)(3 7 6)
(1 5)(2 6 10 9)(4 8)	(1 8 2 7 4)(3 10)(5 9)
(1 10 2 3 4 9 8 6)(5 7)	(1 6 2)(3 5 4 10 8 9)
(1 5 6 10 3 8)(2 4)(7 9)	(1 5 4 7 9)(3 6)
(1 4 10 6 2 7 3 5 8 9)	(1 8)(2 10)(3 5 4 9 6 7)
(1 7)(2 9 4 5)(3 6 8 10)	(2 10 6 5 9 8 4 7 3)
(1 9 4 10 3 6 8 2 5)	(1 6 7 5 4)(2 10 3 8)
(1 8 7)(2 9 10 6)(3 5)	(2 9 7 4 6)(3 10 8 5)
(1 5 4 8 10 2 9 7)(3 6)	(1 7 9 510 6 4)(3 8)
(1 7 6 2 5 4)(3 8)(9 10)	(1 2 8 6 3 10 9 7)

Fourier analysis was used to compute the spectra as per Section 2.1. In this case we used Young's seminormal form of the irreducible representations (see e.g. [6]) to compute the individual Fourier transforms. The sample spacing distribution shown in Figure 6 is representative of the behavior found for each of the generating pairs.

In the case of the cyclic group $\mathbb{Z}/n\mathbb{Z}$, the spectrum can of course be written down exactly. For a pair of integers j, k with $(gcd(j,k), n) = 1$,

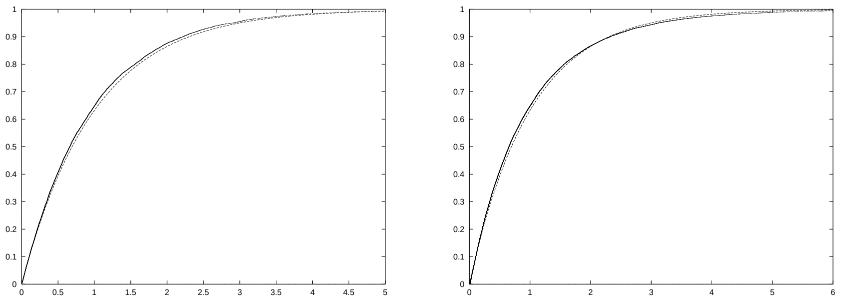

FIG. 6. *$P(S)$ for the Cayley graph on $\mathcal{S}_{10}$ with respect to the generating pair $\{((15)(26109)(48),\ (18274)(310)(59)\}$ (left), and for the Cayley graph on $\mathbb{Z}/n\mathbb{Z}$ with $n = 50,000$ for the generating pair $\{25,693,87,865\}$.*

$(j,k) \neq (1,1)$, the eigenvalues are

$$\lambda_m = \frac{1}{2}\left(\cos\left(\frac{2\pi jm}{n}\right) + \cos\left(\frac{2\pi km}{n}\right)\right) \tag{3.9}$$

for $m = 0, 1, \ldots, n-1$. Choosing $n = 50,000$ we then chose 20 pairs of generators at random and computed the spacing distributions. In each case the resulting spacing distribution is exponential, and Figure 6 is representative of the behavior we have observed.

Together with our calculations for $SL_2(\mathbb{F}_p)$, this is very suggestive that essentially *any* Cayley graph, and perhaps more generally any graph with "sufficiently large" automorphism group, will have Poisson spacings.

4. Summary. We have computed the cdf for the eigenvalue spacings of a variety of Cayley graphs. For graphs (generators) chosen at random we have observed that the spacing distributions are in close agreement with the Poisson distribution. In this spirit we anticipate a "central limit theorem" for random 2-generator Cayley graphs for these groups.

Conjecture 1. *Let $\{G_n\}$ denote any of the three families of groups, S_n, $\mathbb{Z}/n\mathbb{Z}$ or $SL_2(\mathbb{F}_{p_n})$ (for p_n any increasing sequence of primes). For $\epsilon > 0$, as $n \to \infty$ the probability of choosing a pair of generators $\mathcal{S}$ for G_n with $\|P(S) - (1 - e^{-S})\| \geq \epsilon$ goes to zero.*

Remark. Continuing in the spirit of Conjecture 1, it is natural to further conjecture that a random graph with "large" automorphism group will have an adjacency matrix which demonstrates Poisson behavior.

Our previous experiments [9,10] suggested that for $SL_2(\mathbb{F}_p)$, the spectral behavior of any individual Fourier transform for a randomly chosen 2-generator Cayley graph on $SL_2(\mathbb{F}_p)$ is characteristic of the entire graph. Our results here demonstrate a similar relation for the spacings.

Conjecture 2. *Let $X_p = X(SL_2(\mathbb{F}_p), \mathcal{S})$ be a family of k-regular Cayley graphs for $SL_2(\mathbb{F}_p)$. Then asymptotically as $p \to \infty$, the distributions of* *the eigenvalue spacings for X_p and its individual Fourier transforms $\widehat{\delta}_S(\rho)$ are Poisson.*

Remark. All of our computations were performed on HP 735/125 and DEC 3000 Model 600 Alpha workstations, using software written in the C language. The code will be made available via the web page `www.cs.dartmouth.edu/~rockmore/GFT`.

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