Introduction to Global Illumination

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The goal of image synthesis research in computer graphics is the development of methods for modeling and rendering three-dimensional scenes. One of the most challenging tasks of image synthesis is the accurate and efficient simulation of *global illumination* effects: the illumination of objects in a scene by other objects. Early rendering programs shaded surfaces only, not volumes. Furthermore, they treated the visibility (visible surface) and shading tasks independently, employing a *local illumination* model that assumed that the shading of each surface is independent of the shading of other surfaces [Foley et al. 90, p. 760]. Local illumination models typically assume that light comes from a finite set of point light sources only. Global illumination models, on the other hand, recognize that visibility and shading are interrelated: the radiance (roughly, brightness) at one point is determined by the radiances of all other points in the scene.

The physical phenomenon of global illumination is responsible for a number of visual effects: indirect illumination, color bleeding and the *penumbras* or soft shadows from area light sources. Before quantifying global illumination, we discuss some the physics qualitatively.

1 Physics of Light Transport

Light is electromagnetic radiation at wavelengths visible to the human eye. Light interacts with matter in a number of ways. The phenomena most relevant to global illumination are emission, scattering, and absorption. All three of these effects occur in three-dimensional volumes, but because many objects in our world are opaque, it is convenient to speak of surfaces between volumes, and the emission, scattering, and absorption that occurs at surfaces.

Emission can result from various physical processes including incandescence (emission due to heat), luminescence (emission due to chemical reaction, for example), or phosphorescence (time-delayed emission). In image synthesis applications, we typically don't care which mechanism is responsible, we merely need to know the strength of emission as a function of position and direction.

Radiation is absorbed and scattered as it propagates through a medium. A medium is called a *participating medium* if the absorption and scattering are significant, and a *non-participating medium* if they are negligible. In global illumination, for example, we typically regard fog and smoke as participating media, and vacuum or air as non-participating. In scenes with participating media, the emission, scattering, and absorption at each point in space are relevant, so radiant power is a function of three-dimensional position within volumes. In scenes without participating media, the radiation at any point in space is determined by its values at all surface points.

1.1 Scattering, Reflection, and Transmission

In volumes, scattering can be *isotropic*, varying with direction, or *anisotropic*, independent of direction. At a surface, we typically call half of the scattering *reflection* and the other half *transmission*. The fraction of light that is reflected or transmitted in a given direction is called the *reflectance* and *transmittance*, respectively.

If the speed of light changes as light is specularly transmitted from one material into another then the light refracts, or bends. Some materials, such as calcite, are birefringent, having double refraction. The amount of bending varies with the wavelength of light, causing dispersion, the phenomenon responsible for color in rainbows and diamonds.

In this work we make the assumptions of geometric optics, namely that radiation can be simulated using rays and that radiation is *incoherent*, having all phases. These assumptions preclude the simulation of diffraction and interference phenomena. Unless stated otherwise, we also ignore polarization, and assume that scenes are static, media are non-participating, and that surfaces are *gray*, having wavelength-independent properties within the wavelength band of interest. If emission, reflectance, or other properties vary with wavelength then the spectrum can be broken into several wavelength bands. If there is no fluorescence (reflected wavelength different from incident wavelength) then each of these bands can be simulated independently.

2 Terminology and Notation

A derivation of the physics of reflection and transmission for computer graphics is given in [Shirley91]. For an excellent bibliography on the literature of *radiometry* (the study of radiation), see [Horn-Brooks89, bibliography]. A good summary of the basic physics of the related field of thermal radiation is [Eckert-Drake72].

We have tried here to use the standardized terminology of illuminating engineering [ANS86]. This terminology has also been adopted in the physics community [Driscoll78]. In the following, we point out some of the nonstandardized thermal radiation terms as well, but we discourage their use.

The following table summarizes the relevant physical quantities, their dimension, and their units in the metric system:

SYMBOL	PHYSICAL QUANTITY	DIMENSION	UNITS
ρ	reflectance	1	1
τ	transmittance	1	1
λ	wavelength	distance	m
$d\mathbf{x}$	differential surface area	area	m^2
$d\Theta$	differential solid angle	solid angle	steradian
	energy	energy	joule = $\mathrm{kg\,m^2/s^2}$
Φ	power	power = energy/time	watt = joule/s
e	radiant emitted flux density	power/area	watt/m^2
b	radiosity	power/area	watt/m^2
L	radiance	power/(area × solid angle)	watt/(m^2steradian)

In a general form, radiation is a function of phase, polarization, time t, wavelength λ, position $\mathbf{x} = (x, y, z)$, and direction $\Theta = (\theta, \phi)$ [Nicodemus76,Nicodemus78], where θ is polar angle (angle from the surface normal) and ϕ is azimuth.

Radiation on a surface is usually measured in the units of *radiance*, which is defined to be the energy passing through a given area in a given direction in a given amount of time. (This has been called "intensity" in thermal radiation and computer graphics literature, but that term is discouraged because it is not part of the ANSI terminology standard, and it conflicts with the standardized term "radiant intensity" for power per unit solid angle.) The power passing through a differential surface element of radiance L and area $d\mathbf{x}$ at position $\mathbf{x}$, in direction Θ and solid angle $d\Theta$, is

$$\Phi = L(\mathbf{x}, \Theta) \cos\theta \, d\mathbf{x} \, d\Theta$$

The cosine term enters because the projected area of $d\mathbf{x}$ in direction Θ is $d\mathbf{x} \cos\theta$.

2.1 Emission

Emitted radiance integrated over all directions is called *radiant emitted flux density.* (In the thermal radiation field, this quantity is called "emissive power", but use of this term is discouraged because the name suggests erroneously that emissive power has dimension power, when in fact its dimension is power per unit area.) A *diffuse emitter* is one whose emitted radiance is independent of viewing direction.

2.2 Reflection and Transmission

When thermal radiation arrives at a surface, the energy is either reflected back into the original hemisphere, transmitted into the opposite hemisphere, or absorbed (transduced into kinetic energy).

Reflection is fully described by the *bidirectional reflectance distribution function* (BRDF), which is the fraction of energy incident on a surface point from one direction that is reflected in another direction [Nicodemus et al. 77]. If Φ_i is the incident power from incoming direction $\Theta_i = (\theta_i, \phi_i)$ with polar angle θ_i, having solid angle $d\Theta$, and Φ_o is the outgoing power through the same solid angle in direction $\Theta_o = (\theta_o, \phi_o)$, then the bidirectional reflectance is defined as:

$$\rho_{bd}(\Theta_i, \Theta_o) = \frac{\Phi_o}{\Phi_i \cos\theta_i d\Theta}$$

where Θ_i and Θ_o are both directions in the upper hemisphere. Bidirectional transmittance τ_{bd} is defined similarly. Note that the bidirectional reflectance distribution function, at its most general, is a function of four variables: the two dimensions of the input direction and two dimensions of the output direction: $\rho_{bd}(\Theta_i, \Theta_o) = \rho_{bd}(\theta_i, \phi_i, \theta_o, \phi_o)$. Helmholtz' reciprocity law says that the BRDF is symmetric: $\rho_{bd}(\Theta_i, \Theta_o) = \rho_{bd}(\Theta_o, \Theta_i)$.

Many materials have *isotropic* reflectance, for which the azimuthal dependence is a function of the angle between incoming and outgoing azimuth only, reducing the dimension of the BRDF to three:

$$\rho_{bd}(\theta_i, \phi_i, \theta_o, \phi_o) = \rho_{bd}(\theta_o - \theta_i; \phi_i, \phi_o)$$

The most important special cases are *ideal diffuse* or *Lambertian* materials, for which the outgoing radiance is direction-independent, and *ideal specular* materials, for which the outgoing radiance is limited to the solid angle of the incident radiation around the mirror

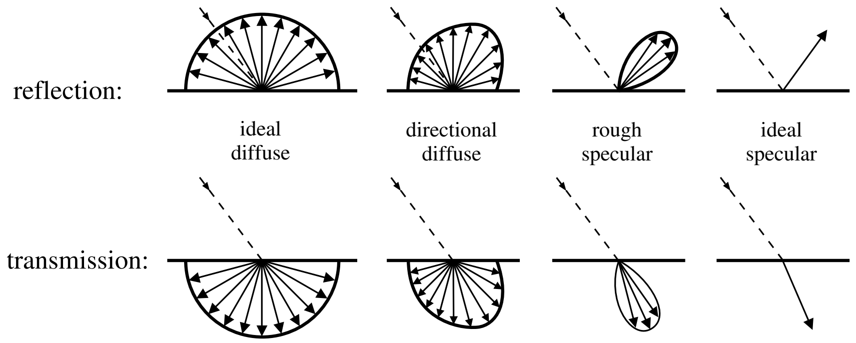

Figure 1: *Four classes of reflectance and transmittance: ideal diffuse, directional diffuse, rough specular, and ideal specular; showing a polar plot of radiance for fixed incoming direction and varying outgoing direction.*

direction (figure 1). Chalk and polished metal are approximations of ideal diffuse and ideal specular reflectors, and rice paper and glass are approximations of ideal diffuse and ideal specular transmitters, respectively. The BRDF of a diffuse material is constant, while that of a specular material is a delta function:

$$\rho_{\text{spec}}(\theta_i, \phi_i, \theta_o, \phi_o) = \rho_0 \delta(\theta_o - \theta_i) \delta(\phi_o - \phi_i - \pi \bmod 2\pi)$$

Real materials have neither of these idealized reflectance distributions. Brushed metal, for example, is anisotropic. Attempts have been made to approximate empirical BRDF's mathematically both in the thermal radiation [Torrance-Sparrow67] and computer graphics [Phong75,Blinn77,Cook-Torrance82,He et al. 91] communities. Approximations good enough to make very realistic-looking images are possible by modeling a rough surface as a piecewise smooth surface with random microscopic bumps.

In addition to ideal diffuse and ideal specular reflectance, it is helpful to define two additional classes. We define *directional diffuse* to be reflection that is a smooth but non-constant function of direction [Sillion et al. 91], and *rough specular* to be reflection to a finite number of cones [Heckbert90]. (These two classes overlap, and their definitions are intentionally vague, in order that they and the two ideal classes will cover all possible BRDF's.) Directional diffuse and rough specular transmittance are defined analogously.

An ideal diffuse surface has equal radiance from all viewing directions, but a general surface's radiance varies with viewing direction, so we say that ideal diffuse reflection is *view-independent* while general reflection is *view-dependent*. For computer graphics purposes, the simplest materials have a position-invariant, isotropic BRDF consisting of a linear combination of ideal diffuse and ideal specular reflection, but a fully-general BRDF can be position-dependent and simulate textured, anisotropic, directional diffuse or rough specular surfaces.

We define *scattering* to mean either reflection or transmission. The BRDF ρ_{bd} and BTDF τ_{bd} can be regarded as two halves of a *bidirectional scattering distribution function* (BSDF) which we denote ρ_{bds}. Merging the reflectance and transmittance functions into one simplifies the notation. For example, when computing the outgoing radiance of light, two integrals are needed if the BRDF and BTDF are used (one for each hemisphere) but if the BSDF is used, only a single integral (over an entire sphere) is needed.

2.3 Diffuse Reflectance

The fraction of the incident radiance for a given incoming direction that is reflected anywhere can be calculated by integrating the bidirectional reflectance over the hemisphere of outgoing directions, yielding the *diffuse reflectance*:

$$\rho_d(\Theta_i) = \int_{\text{hemi}} d\Theta_o \cos\theta_o \, \rho_{bd}(\Theta_i, \Theta_o)$$

For an diffuse material, ρ_{bd} is constant, and $\rho_d = \pi\rho_{bd}$. Diffuse transmittance τ_d is defined analogously.

The fraction of radiation from direction Θ_i that is absorbed is denoted $\alpha(\Theta_i)$. This is the *absorptance*. The fractions of the incident radiance with incoming direction Θ_i that are reflected, transmitted, and absorbed are therefore $\rho_d(\Theta_i)$, $\tau_d(\Theta_i)$, and $\alpha(\Theta_i)$, respectively. By conservation of energy, they sum to one at each position and in each direction: $\rho_d(\Theta_i) + \tau_d(\Theta_i) + \alpha(\Theta_i) = 1$. Since radiance is nonnegative, each of these coefficients must be greater than or equal to zero. In practice, zero reflectance and transmittance are never achieved, so $0 < \rho_d,\ \tau_d,\ \alpha < 1$. Because of dirt and other factors, it is uncommon to find diffuse reflectances above .85 [Ward90].

3 Why Integral Equations?

Global illumination is governed by an integral equation. Why this comes about and what this means are described intuitively here, then a brief tutorial on integral equations is given, and the integral equation governing global illumination is discussed. The following discussion is specialized to the case of non-participating media.

The light leaving a surface consists of two parts, the emitted light (if this is the surface of a light source) and the reflected or transmitted light. In general, the radiance of a surface is a function of both position and viewing direction.

Assuming that the geometry and the emissive, reflective and transmissive properties of all surfaces are known, the radiance at each surface point in each direction is determined. The light scattered at a surface point in a given direction could, in general, have come from any direction. Some of this light will come from the designated light sources, and other light will come from other surfaces. The radiance at a point is thus related to the radiance of all points visible to it.

The equation describing the interdependency thus has the general form

$$\text{radiance}(p) = \text{emit}(p) + \int_{\texttt{all surfaces}} \text{radiance}(q) \, \text{scat}(p,q) \; dq$$

where p and q each represent a point and direction, where radiance(p) is the outgoing radiance at p, where emit(p) is the emitted radiance at p, and scat(p,q) is the fraction of light leaving q that is scattered to p. Naturally, scat$(p,q) = 0$ if there is an occluding object between points p and q. The above is called an integral equation because the unknown radiance function appears inside an integral. When the radiation passes through a participating medium, the equation becomes *integro-differential*: a combination of both integrals and differentials.

4 Integral Equations

There are a number of good books on integral equations for both the beginner [Jerri85] and the mathematically sophisticated [Delves-Mohamed85,Atkinson76,Courant-Hilbert37, Hildebrand65]. Most of this section is adapted from [Delves-Mohamed85,Atkinson76].

An *integral equation* is an equation in which the function to be determined appears inside an integral. The class of integral equation of interest in this work is the *Fredholm integral equation of the second kind*, with general form[1]

$$b(s) = e(s) + \int_{\alpha}^{\beta} dt\, \kappa(s,t) b(t)$$

where e and the *kernel* κ are given and b is to be determined. This is a Fredholm integral equation because the limits of integration α and β are constant, and it is a second-kind integral equation because the unknown function $b(s)$ appears outside the integral. Most integral equation properties and solution methods generalize when the domain variables s and t are multi-dimensional [Courant-Hilbert37, p. 152].

The above equation is abbreviated as $b = e + \mathcal{K}b$, where $\mathcal{K}$ denotes the integral operator which when applied to a function b, yields a function

$$(\mathcal{K}b)(s) = \int_{\alpha}^{\beta} dt\, \kappa(s,t) b(t)$$

We write the inner product of two real functions f and g over the domain $[a,b]$ as

$$(f,g) = \int_{\alpha}^{\beta} ds\, f(s) g(s)$$

Functions f and g are said to be *orthogonal* if $(f,g) = 0$.

Formally, our integral equation $b = e + \mathcal{K}b$ can be rewritten $(\mathcal{I} - \mathcal{K})b = e$, where $\mathcal{I}$ is the identity operator, and the solution can be obtained by inverting the operator $\mathcal{I} - \mathcal{K}$:

$$b = (\mathcal{I} - \mathcal{K})^{-1} e$$

It is typically impractical and unnecessary to invert the integral operator explicitly, however [Alpert90].

4.1 Neumann Series Approximation

Approximate solutions to many integral equations can be found iteratively. Starting with some initial guess $b^{(0)}(s)$, subsequent approximations are defined by

$$b^{(i)} = e + \mathcal{K} b^{(i-1)}$$

If we start with $b^{(0)} = e$, then the ith approximant is the truncated series

$$b^{(i)} = e + \mathcal{K}e + \mathcal{K}^2 e + \cdots + \mathcal{K}^i e$$

where $\mathcal{K}^i$ denotes i successive applications of the integral operator $\mathcal{K}$.

[1] We use the notation "$\int dx\, f(x)$" instead of "$\int f(x)\, dx$" because it most clearly indicates the variable of integration during multiple integration.

The sequence $b^{(i)}$ converges if the norm of the integral operator is less than 1 ($||\mathcal{K}|| < 1$), where the operator norm is defined in terms of a function norm:

$$||\mathcal{K}|| = \max_{b \neq 0} \frac{||\mathcal{K}b||}{||b||} \tag{1}$$

The L_p norms are a general class of function norms:

$$||f||_p = \left(\int_\alpha^\beta dx\, |f(x)|^p \right)^{1/p} \tag{2}$$

the most common of which are the L_1 norm, the L_2 norm, and the L_∞ norm.

When $||\mathcal{K}|| < 1$, the exact solution to the integral equation is given by the *Neumann series*

$$b = b^{(\infty)} = \sum_{i=0}^{\infty} \mathcal{K}^i e \tag{3}$$

The Neumann series is a generalization of the geometric series for $1/(1-\alpha)$.

5 The Global Illumination Equation

We derive the integral equation defining radiance in a scene with non-participating media and opaque surfaces with general reflectance [Özisik73].

We now make explicit the position-dependence of the surface properties. At a surface, the outgoing radiance L_o at position $\mathbf{x}$ in direction Θ_o equals the emitted radiance L_{emit}, plus the scattered radiance L_{scat}:

$$L_o(\mathbf{x}, \Theta_o) = L_{\text{emit}}(\mathbf{x}, \Theta_o) + L_{\text{scat}}(\mathbf{x}, \Theta_o)$$

The scattered radiance in a given outgoing direction is the integral of the bidirectional scattering distribution function times the incident radiance over all incoming directions:

$$L_{\text{scat}}(\mathbf{x}, \Theta_o) = \int_{\text{sphere}} d\Theta_i \cos\theta_i\, \rho_{bds}(\mathbf{x}, \Theta_i, \Theta_o)\, L_i(\mathbf{x}, \Theta_i) \tag{4}$$

So the outgoing radiance function is $L_o = L_{\text{emit}} + L_{\text{scat}}$ [Özisik73, eq. 4-1]:

$$L_o(\mathbf{x}, \Theta_o) = L_{\text{emit}}(\mathbf{x}, \Theta_o) + \int_{\text{sphere}} d\Theta_i \cos\theta_i\, \rho_{bds}(\mathbf{x}, \Theta_i, \Theta_o)\, L_i(\mathbf{x}, \Theta_i) \tag{5}$$

In a non-participating medium, radiance is not attenuated with distance, so if the first surface point hit by the ray from point $\mathbf{x}$ in direction Θ_i is $\mathbf{x}'$, and the polar angle and azimuth of this ray are $\Theta'_o = (\theta'_o, \phi'_o)$ in the local coordinate system of point $\mathbf{x}'$, then the incoming radiance at $\mathbf{x}$ and the outgoing radiance at $\mathbf{x}'$, in this common direction are equal (figure 2):

$$L_i(\mathbf{x}, \Theta_i) = L_o(\mathbf{x}', \Theta'_o)$$

By changing the variable of integration from incoming direction Θ_i to surface position $\mathbf{x}'$, we can eliminate the incoming radiance function, leaving the outgoing radiance function as the only unknown. To do this we find the solid angle $d\Theta_i$ subtended by a surface element

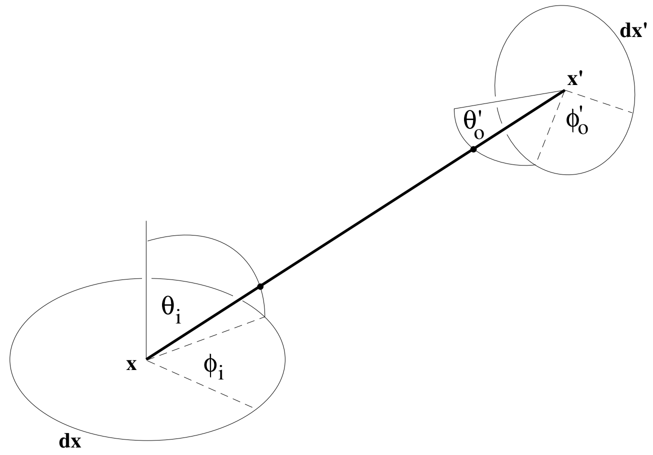

Figure 2: *Geometry of thermal radiation leaving point* $\mathbf{x}'$ *and arriving at point* $\mathbf{x}$.

with area $d\mathbf{x}'$ at a distance r and angle θ'_o to the normal at $\mathbf{x}'$. The solid angle of $d\mathbf{x}'$ at distance r equals this area projected onto a sphere of radius r, divided by r^2, so:

$$d\Theta_i = \frac{d\mathbf{x}' \cos\theta'_o}{r^2}$$

Therefore, the integral equation governing global illumination in a scene with general emission and scattering in a non-participating medium is[2]:

$$L_o(\mathbf{x}, \Theta_o) = L_{\text{emit}}(\mathbf{x}, \Theta_o) + \int_\Gamma d\mathbf{x}' \frac{\cos\theta_i \cos\theta'_o}{r^2} \, v \, \rho_{bds}(\mathbf{x}, \Theta_i, \Theta_o) \, L_o(\mathbf{x}', \Theta'_o) \tag{6}$$

where v is the *visibility function*, which equals 1 if $\mathbf{x}$ and $\mathbf{x}'$ are inter-visible, and 0 if they are occluded from each other's view; and Γ is the set of all surfaces in the scene. The variables Θ_i, Θ'_o, r, and v are functions of $\mathbf{x}$ and $\mathbf{x}'$. The only unknown in this equation is the outgoing radiance L_o.

We call this the *global illumination equation* for non-participating media. It is a Fredholm integral equation of the second kind.

Integral equations similar to this have appeared in the thermal radiation literature [Özisik73], in illuminating engineering [Moon36], in the field of neutron transport in physics, where it is called the *Boltzmann equation* [Lewis-Miller84], in computer vision [Koenderink-van Doorn83], where it has been called the *mutual illumination equation* [Forsyth-Zisserman89], and in computer graphics, where it has been called the *rendering equation* [Kajiya86,Immel et al. 86,Bouville et al. 90,Shirley91].

5.1 Integral Equation for Diffuse, Opaque Surfaces

The equation simplifies if we assume that all surfaces are opaque, diffuse emitters and reflectors. Then radiant emitted flux density and reflectance are functions of position only, and we can substitute hemispherical quantities: the radiant emitted flux density $e(\mathbf{x}) = \pi L_{\text{emit}}(\mathbf{x}, \Theta_o)$, the diffuse reflectance $\rho_d(\mathbf{x}) = \pi \rho_{bds}(\mathbf{x}, \Theta_i, \Theta_o)$, and the *radiosity* $b(\mathbf{x}) =$

[2] a generalization of Özisik's equations 4-1 and 5-1.

$\pi L_o(\mathbf{x}, \Theta_o)$. Radiosity[3] is the sum of emitted and reflected radiation over the hemisphere of directions. A more standard term for "radiosity" is "radiant exitance", but we will use the former for historical reasons.

With diffuse reflectance, the radiance of outgoing radiation is independent of the directional distribution of incident radiation: $L_o(\mathbf{x}, \Theta) = L_o(\mathbf{x})$. Put another way, diffuse reflectance obliterates the history of the incident radiation [Sparrow-Cess78]. In a non-participating medium, radiance is then a function of two-dimensional surface position only.

The integral equation governing diffuse, opaque surfaces in a non-participating medium is:

$$b(\mathbf{x}) = e(\mathbf{x}) + \rho_d(\mathbf{x}) \int_\Gamma d\mathbf{x}' \frac{\cos\theta_i \cos\theta'_o}{\pi r^2}\, v\, b(\mathbf{x}')$$

Essentially equivalent integral equations are given in [Sparrow-Haji-Sheikh65], [Hottel-Sarofim67, eq. 3-6a], [Özisik73, eq. 5-1].

5.2 Properties of the Integral Equation of Global Illumination

The kernel of this integral equation in a specular scene is non-smooth and sparse (zero almost everywhere), while the kernel in a diffuse scene with no occlusions is smooth and dense (nonzero almost everywhere). The special case of global illumination in two-dimensional, diffuse scenes is discussed in depth elsewhere [Heckbert91,Heckbert92].

Global illumination algorithms can be characterized by the approximations they make to the global illumination equation. In diffuse scenes, one can exploit the smoothness of the kernel using radiosity algorithms, and in specular scenes, one can exploit the sparseness of the kernel using ray tracing algorithms. These solution methods are explored in succeeding chapters.

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[3] The word "radiosity" was coined by Parry Moon [Moon36]. Moon invented a number of other radiometric terms such as "pharosage" and "helios" that never caught on. Though the word is popular today, at least in computer graphics, "radiosity" was called "an undesirable word" by Hoyt Hottel [Hottel-Sarofim67, p. 74], one of the people who in the 50's invented the discretization technique that later became known as the "radiosity method" [Sparrow63].

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