CG
11/11/10

Graph Rigidity

Def G is rigid in $\mathbb{R}^2$ if some embedding is infinitesimally rigid.

Laman: G is minimally rigid iff

1) $m = 2n-3$

2) $\forall H \subseteq G \quad m' \leq 2n'-3$

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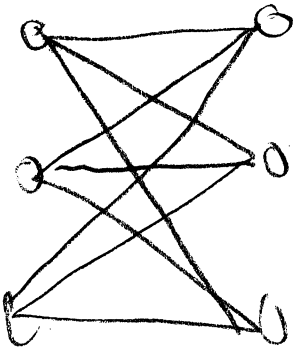
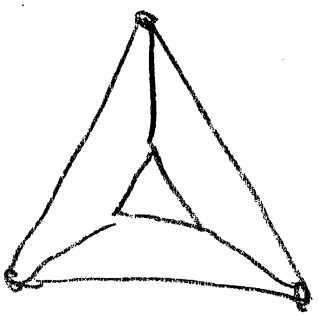
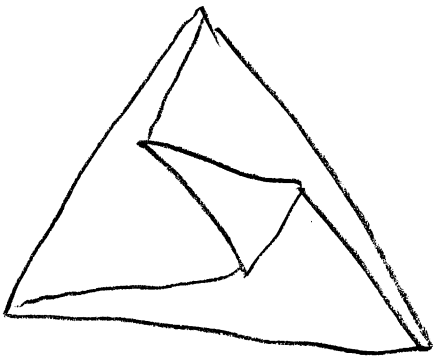
eg $K_{3,3}$



Prism



not inf-rigid



inf-rigid

<u>Def</u> G is Infinitesimally rigid if

$\forall V = (v_1 \cdots v_n)$ $v_i \in \mathbb{R}^2$ if

$(v_i - v_j) \cdot (p_i - p_j) = 0 \quad \forall (i,j) \in E$

then $V \equiv 0$

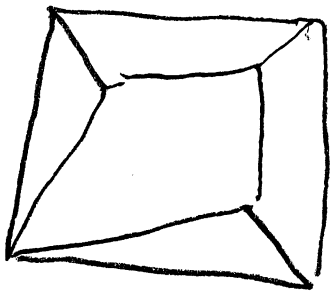
<u>Def</u> G is minmally rigid if

G is inf-rigid & $G \setminus e$ is not $\forall e \in E$.

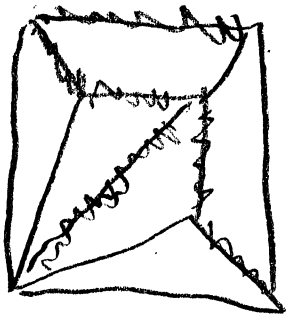
Lovász-Yemini 82: G is minimally rigid iff G + edge $\equiv$ 2 edge disjoint spanning trees.

eg

G



G + edge



Henneberg 1860: G is min rigid iff

G can be obtained using Henneberg 2 rules

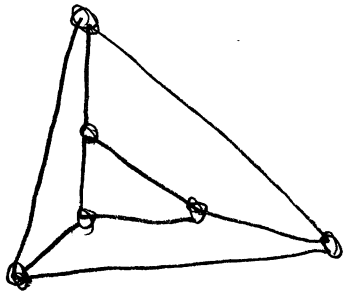
<u>Rule 0</u> start with an edge.

<u>Rule 1</u> Add vertex of degree 2

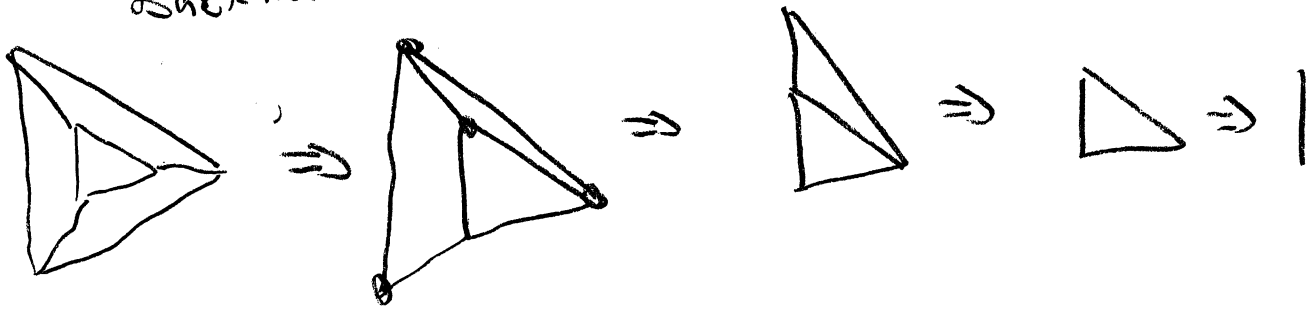
<u>Rule 2</u> 1) remove (v,w)

2) add x adj to $v, w, z \in V$

<u>eg</u>



backwards



<u>Claim</u> T embedded graph $m < 2n-3$ then T is not rigid!

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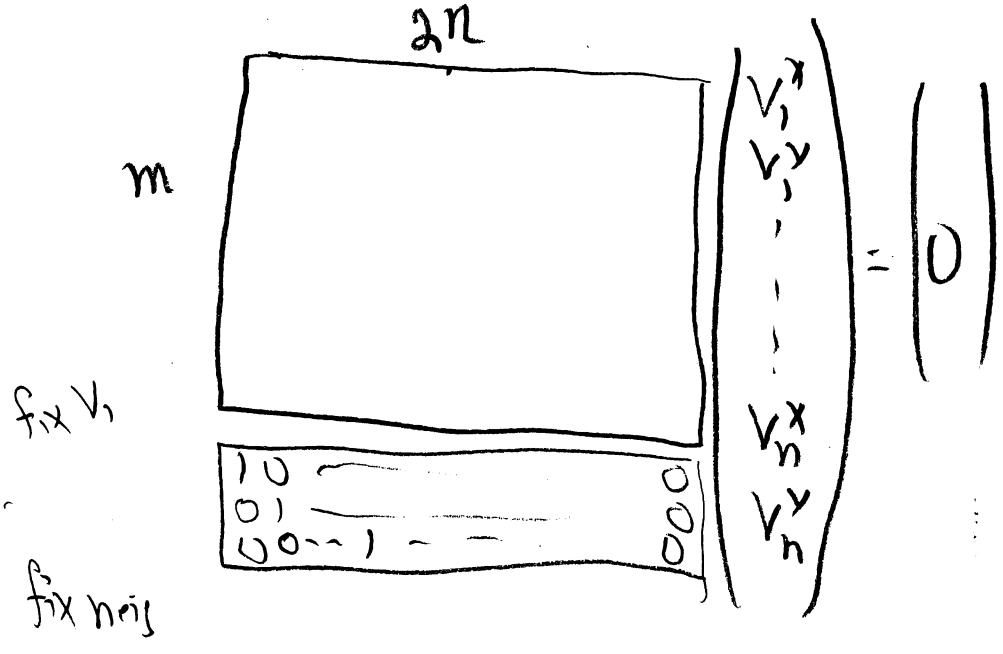
ie $\exists\, v = (v_1 \cdots v_n)$ not a rigid motion

$$(v_i - v_j)\cdot(P_i - P_j) = 0 \quad \forall (i,j) \in E$$



$m+3 < 2n$ done!

<u>Lemma</u> G is Laman then G has a Henneberg construction.

pf Induction on n. $n>3$

$m=2n-3$ $\exists v \in V$ of degree ≤ 3

If degree$(v)=2$ remove it

Then $G \setminus v$ is Laman

If $d(v)=3$ then $\underset{G'}{\underset{\shortparallel}{G \setminus v}}$ $m'=2n'-4$

then G' is not rigid

Compute maximal rigid components of G'

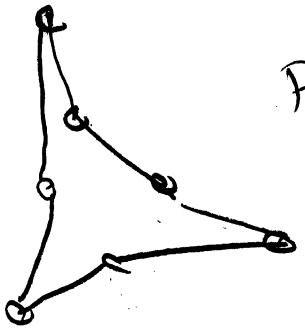
$G_1' \cdots G_x'$

v must attach to at least 2 G_i'.

Else $G_i' \cup (v)$ has too many edges.

Pseudo-Tri & Rigidity

Pseudo-tri examples



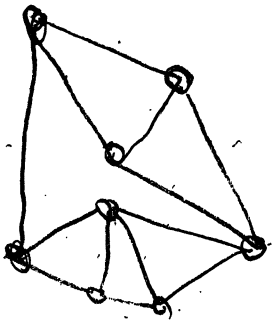
Pseudo-Δ

Vertex types
convex - corners
straight
reflex

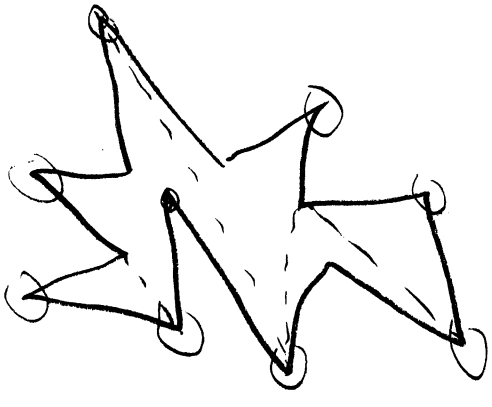
inside is P-3-gon
outside is P-4-gon

Pseudo-triangulation (including CH)

P-Δ



Triangulating a Polygon



Suppose G is embedded planar.

<u>Def</u> A vertex is <u>pointed</u> if some angle at x is reflex.

G is <u>pointed</u> if all $v \in V$ are pointed

<u>Lemma</u> If T is pointed P-Δ of n points P then

$$m = |E(T)| = 2n-3$$

$t \equiv \#\Delta$ in T $\quad m = \#$edges

Euler: $t - m + n = 1$ (no outside face)

$\#$triangles $= 2m$

$\#$ corners $\equiv 2m-n$ (pointed condition)

$$3t \equiv 2m-n$$

Euler $\quad 3t - 3m + 3n = 3$

$$2m - n - 3m + 3n = 3$$

$$-m + 2n = 3$$

$$m = 2n-3$$

<u>Thm</u> T is pointed P-D of a CH then T is rigid.

We consider dual problem!

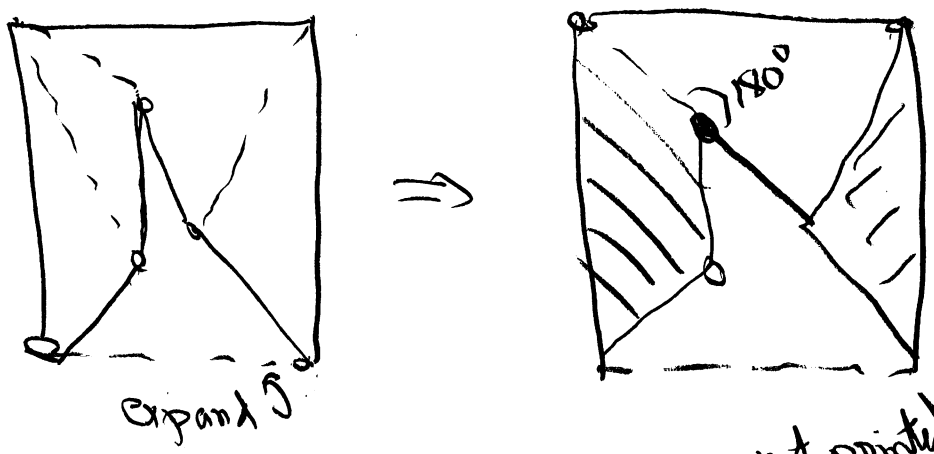
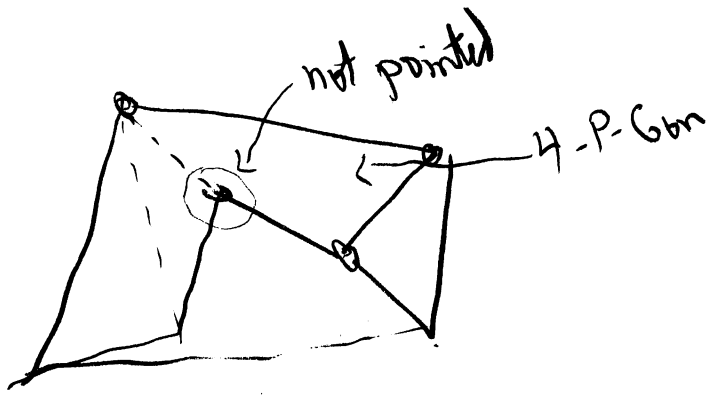
<u>Maxwell lifting</u>

<u>Claim</u> If f is a lifting of a face F & P is a reflex vertex of F then

$P \in \mathrm{Max}_f(F)$ iff f on F is constant.

<u>Thm</u> If G is a pointed P-Δ & $e \in CH(G)$ then expanding e gives an expansive motion of G.

Convexifying a Polygon

<u>Alg</u> (R polygon)

1) Add struts to R forming pointed PD (convex)
2) While ∃ strut s on CH do
   a) expand s until some pair e & e' are straight
      i) if e & e' ∈ Bars then freeze e & e'
      else clean up by edge flipping