Spectral Partitioning Works: Planar graphs and finite element meshes*

Daniel A. Spielman†
Department of Mathematics
Massachusetts Institute of Technology
Cambridge, MA 02139
spielman@math.mit.edu

Shang-Hua Teng‡
Department of Computer Science
University of Minnesota
Minneapolis, MN 55455
steng@cs.umn.edu

Abstract

Spectral partitioning methods use the Fiedler vector—the eigenvector of the second-smallest eigenvalue of the Laplacian matrix—to find a small separator of a graph. These methods are important components of many scientific numerical algorithms and have been demonstrated by experiment to work extremely well. In this paper, we show that spectral partitioning methods work well on bounded-degree planar graphs and finite element meshes—the classes of graphs to which they are usually applied. While naive spectral bisection does not necessarily work, we prove that spectral partitioning techniques can be used to produce separators whose ratio of vertices removed to edges cut is $O(\sqrt{n})$ for bounded-degree planar graphs and two-dimensional meshes and $O(n^{1/d})$ for well-shaped d-dimensional meshes. The heart of our analysis is an upper bound on the second-smallest eigenvalues of the Laplacian matrices of these graphs: we prove a bound of $O(1/n)$ for bounded-degree planar graphs and $O(1/n^{2/d})$ for well-shaped d-dimensional meshes.

*This abstract is based on U.C. Berkeley Technical Report UCB//CSD-96-898

†`http://www-math.mit.edu/~spielman`. Supported in part by an NSF postdoc.

‡`http://www.cs.umn.edu/~steng`. Supported in part by an NSF CAREER award (CCR-9502540) and an Intel research grant. Part of the work was done while visiting Xerox PARC.

1. Introduction

Spectral partitioning has become one of the most successful heuristics for partitioning graphs and matrices. It is used in many scientific numerical applications, such as mapping finite element calculations on parallel machines [Sim91, Wil90], solving sparse linear systems [PSW92], and partitioning for domain decomposition [CR87, CS93]. It is also used in VLSI circuit design and simulation [CSZ93, HK92, AK95]. Substantial experimental work has demonstrated that spectral methods find good partitions of the graphs and matrices that arise in many applications [BS92, HL92, HL93, PSL90, Sim91, Wil90]. However, the quality of the partition that these methods should produce has so far eluded precise analysis. In this paper, we will prove that spectral partitioning methods give good separators for the graphs to which they are usually applied.

The size of the separator produced by spectral methods can be related to the Fiedler value—the second smallest eigenvalue of the Laplacian—of the adjacency structure to which they are applied. By showing that well-shaped meshes in d dimensions have Fiedler value at most $O(1/n^{2/d})$, we show that spectral methods can be used to find bisectors of these graphs of size at most $O(n^{1-1/d})$. While a small Fiedler value does not immediately imply that there is a cut along the Fiedler vector that is a balanced separator, it does mean that there is a cut whose ratio of vertices separated to edges cut is $O(n^{1/d})$. By removing the vertices separated by this cut, computing a Fiedler vector of the new graph, and iterating as necessary, one can find a bisector of $O(n^{1-1/d})$ edges. In particular, we prove that maximum-degree Δ planar graphs have Fiedler value at most $8\Delta/n$, which implies that spectral techniques can be used to find bisectors of size at most $O(\sqrt{n})$ in these graphs. These bounds are the best possible for well-shaped meshes and planar graphs.

1.1. History

The spectral method of graph partitioning was born in the works of Donath and Hoffman [DH72, DH73] who first suggested using the eigenvectors of adjacency matrices of graphs to find partitions. Fiedler [Fie73, Fie75a, Fie75b] associated the second-smallest eigenvalue of the Laplacian of a graph with its connectivity and suggested partitioning by splitting vertices according to their value in the corresponding eigenvector. Thus, we call this eigenvalue the *Fiedler value* and a corresponding vector a *Fiedler vector*.

A few years later, Barnes and Hoffman [Bar82, BH84] used linear programming in combination with an examination of the eigenvectors of the adjacency matrix of a graph. In a similar vein, Boppana [Bop87] analyzed eigenvector techniques in conjunction with convex programming. However, the use of linear and convex programming made these techniques impractical for most applications.

By recognizing a relation between the Fiedler value and the Cheeger constant [Che70] of continuous manifolds, Alon [Alo86] and Sinclair and Jerrum [SJ89] demonstrated that if the Fiedler value of a graph is small, then directly partitioning the graph according to the values of vertices in the eigenvector will produce a cut with a good ratio of cut edges to separated vertices (see also [AM85, Fil91, DS91, Mih89, Moh89]). Around the same time, improvements in algorithms for approximately computing eigenvectors, such as the Lanczos algorithm, made the computation of eigenvectors practical [PSS82, Sim91]. In the next few years, a wealth of experimental work demonstrated that spectral partitioning methods work well on graphs that usually arise in practice [BS92, HL92, PSL90, Sim91, Wil90]. Still, researchers were unable to prove that spectral partitioning techniques would work well on the graphs encountered in practice. This failure is partially explained by results of Guattery and Miller [GM95] demonstrating that naive applications of spectral partitioning, such as spectral bisection, will fail miserably on some graphs that could conceivably arise in practice. By bounding the Fiedler values of the graphs of interest in scientific applications—bounded-degree planar graphs and well-shaped meshes—we are able to show that spectral partitioning methods will successfully find good partitions of these graphs.

In a related line of research, algorithms were developed along with proofs that they will always find small separators in various families of graphs. The seminal work in this area was that of Lipton and Tarjan [LT79], who constructed a linear-time algorithm that produces a 1/3-separator of $\sqrt{8n}$ nodes in any n-node planar graph. Their result improved a theorem of Ungar [Ung51] which demonstrated that every planar graph has a separator of size $O(\sqrt{n}\log n)$. Gilbert, Hutchinson, and Tarjan [GHT84] extended these results to show that every graph of genus at most g has a separator of size $O(\sqrt{gn})$. Another generalization was obtained by Alon, Seymour, and Thomas [AST90], who showed that graphs that do not have an h-clique minor have separators of $O(h^{3/2}\sqrt{n})$ nodes. Plotkin, Rao, and Smith [PRS94] reduced the dependency on h from $h^{3/2}$ to h. Using geometric techniques, Miller, Teng, Thurston, and Vavasis [MT90, MTTV96a, MTTV96b, MTV91, MV91, Ten91] extended the planar separator theorem to graphs embedded in higher dimensions and showed that every well-shaped mesh in R^d has a $1/(d+2)$-separator of size $O(n^{1-1/d})$. Using multicommodity flow, Leighton and Rao [LR88] designed a partitioning method guaranteed to return a cut whose ratio of cut size to vertices separated is within logarithmic factors of optimal. While spectral methods have been favored in practice, they lacked a proof of effectiveness.

1.2. Outline of paper

In Section 2, we introduce the concept of a graph partition, review some facts from linear algebra that we require, and describe the class of spectral partitioning methods. In Section 3, we prove the *embedding lemma*, which relates the quality of geometric embeddings of a graph with its Fiedler value. We then show (using the main result of Section 4) that every planar graph has a "nice" embedding as a collection of spherical caps on the surface of a unit sphere in three dimensions. By applying the embedding lemma to this embedding, we prove that the Fiedler value of every bounded-degree planar graph is $O(1/n)$. In Section 4, we show that, for almost every arrangement of spherical caps on the unit sphere in R^d, there is a sphere-preserving map that transforms the caps so that the center of the sphere is the centroid of their centers. It is this fact that enables us to find nice embeddings of planar graphs. In Section 5, we extend our spectral planar separator theorem to well-shaped meshes. This extension enables us to show that the spectral method finds cuts of ratio $O(1/n^{1/d})$ for k-nearest neighbor graphs and well-shaped finite element meshes. In the full paper, we extend the results of Guattery and Miller to show that our results are essentially the best possible given current characterizations of well-shaped meshes. We present natural families of graphs for which Fiedler vectors can be used to find cuts of good ratio, but not good balance. We discuss why these graphs exist and why they might not appear in practice.

2. Introduction to Spectral Partitioning

In this section, we define the spectral partitioning method and introduce the terminology that we will use throughout the paper.

2.1. Graph Partitioning

Throughout this paper, $G = (V, E)$ will be a connected, undirected graph on n vertices.

A *partition* of a graph G is a division of its vertices into two disjoint subsets, A and $\bar{A}$. Without loss of generality, we can assume that $|A| \leq |\bar{A}|$. Let $E(A, \bar{A})$ be the set of edges with one endpoint in A and the other in $\bar{A}$. The *cut size* of the partition $(A, \bar{A})$ is simply $|E(A, \bar{A})|$. The *ratio* of the cut, denoted $\phi(A, \bar{A})$, is equal to the ratio of the size of the cut to the size of A, namely,

$$\phi(A, \bar{A}) = \frac{|E(A, \bar{A})|}{\min(|A|, |\bar{A}|)}.$$

The isoperimetric number of a graph, which measures how good a cut ratio one can hope to find, is defined to be

$$\phi(G) = \min_{|A| \leq n/2} \frac{|E(A, \bar{A})|}{|A|}.$$

A partition is a *bisection* of G if A and $\bar{A}$ differ in size by at most 1. Given an algorithm that can find cuts of ratio ϕ in G and its subgraphs, we can find a bisector of G of size $O(\phi n)$.

Lemma 1. *Assume that we are given an algorithm that will find a cut of ratio at most $\phi(k)$ in every k-node subgraph of G, for some monotonically decreasing function ϕ. Then repeated application of this algorithm can be used to find a bisection of G of size at most*

$$\int_{x=1}^{n} \phi(x) dx.$$

Proof: The following algorithm (see [LT79, Gil80]) will find the bisection.

i. Initially, let $D^{(0)} = G$, let A and B be empty sets, and let $i = 0$.

ii. If $D^{(i)}$ is empty, then return A and B; otherwise repeat

   (a) Find a cut of ratio at most $\phi(|D^{(i)}|)$ that divides $D^{(i)}$ into $F^{(i)}$ and $\overline{F^{(i)}}$. We assume that $|F^{(i)}| \leq |\overline{F^{(i)}}|$.

   (b) If $|A| \leq |B|$, let $A = A \cup F^{(i)}$; otherwise, let $B = B \cup F^{(i)}$;

   (c) Let $D^{(i+1)} = \overline{F^{(i+1)}}$, let $i = i + 1$, and return to step (a).

We assume that the algorithm terminates after t iterations. To show that this algorithm produces a bisection, we need to prove that, for all i in the range $0 \leq i < t$, $\min(|A|, |B|) + |F^{(i)}| \leq n/2$. Because $|F^{(i)}| \leq |\overline{F^{(i)}}|$,

$$\min(|A|, |B|) + |F^{(i)}| \leq (|A| + |B| + |F^{(i)}| + |\overline{F^{(i)}}|)/2 = n/2.$$

We now analyze the total cut size. Because the algorithm finds cuts of ratio at most $\phi(|D^{(i)}|)$ at the ith iteration, the number of edges we cut to separate $F^{(i)}$ is at most

$$\begin{aligned}
\phi(|D^{(i)}|)|F^{(i)}| &= \sum_{j=1}^{|F^{(i)}|} \phi(|D^{(i)}|) \\
&= \sum_{j=|D^{(i)}|}^{|D^{(i)}|-|F^{(i)}|+1} \phi(|D^{(i)}|) \\
&\leq \sum_{j=|D^{(i)}|}^{|D^{(i)}|-|F^{(i)}|+1} \phi(j)
\end{aligned}$$

The inequality follows from the fact that ϕ is monotonicly decreasing. The total number of edges cut by this algorithm is at most

$$\begin{aligned}
\sum_{i=0}^{t-1} \phi(|D^{(i)}|)|F^{(i)}| &\leq \sum_{i=0}^{t-1} \left(\sum_{j=|D^{(i)}|}^{|D^{(i)}|-|F^{(i)}|+1} \phi(j) \right) \\
&= \sum_{j=1}^{n} \phi(j) \\
&\leq \int_{1}^{n} \phi(x) dx
\end{aligned}$$

The last inequality follows from the assumption that ϕ is monotonically decreasing. $\square$

Remark 2. *If $\phi(x) = x^{-1/d}$ then*

$$\int_{1}^{n} \phi(x) dx = \frac{d}{d-1}(n^{1-1/d} - 1).$$

Lipton and Tarjan [LT79] showed that by repeatedly applying an α-separator of size $\beta\sqrt{n}$, one can obtain a bisection of size $\beta/(1-\sqrt{1-\alpha})\sqrt{n}$. Gilbert [Gil80] extended this result to graphs with positive vertex weights at the expense of a $1/(1-\sqrt{2})$ factor in the bisection bound. Djidjev and Gilbert [DG92] further generalized this result to graphs with arbitrary weights. Leighton and Rao [LR88] showed that one can obtain an $O(\alpha)$-approximation to a 1/3-separator from an α-approximation to a ratio cut.

2.2. Laplacians and Fiedler Vectors

The adjacency matrix, $A(G)$, of a graph G is the $n \times n$ matrix whose (i,j)-th entry is 1 if $(i,j) \in E$ and 0 otherwise. The diagonal entries are defined to be 0. Let D be the $n \times n$ diagonal matrix with entries $D_{i,i} = d_i$, where d_i is the degree of the ith vertex of G. The *Laplacian*, $L(G)$, of the graph G is defined to be $L(G) = D - A$.

Notice that the all-ones vector is an eigenvector of any Laplacian matrix and that its associated eigenvalue is 0. Because Laplacian matrices are positive semidefinite, all their other eigenvalues must be non-negative. We will focus on the second smallest eigenvalue, λ_2, of the Laplacian and an associated eigenvector $\vec{u}$. Fiedler called this eigenvalue the "algebraic connectivity of a graph", so we will call it the *Fiedler value* and an associated eigenvector a *Fiedler vector*.

For any vector $\vec{x} \in R^n$, we have

$$\vec{x}^T L(G)\vec{x} = \sum_{(i,j)\in E} (x_i - x_j)^2.$$

Moreover, the Fiedler value, λ_2, of G is given by

$$\lambda_2 = \min_{\vec{x}\perp(1,1,\ldots,1)} \frac{\vec{x}^T L(G)\vec{x}}{\vec{x}^T\vec{x}},$$

with the minimum occurring only when $\vec{x}$ is a Fiedler vector. In general, the *Rayleigh quotient* of a vector $\vec{x}$ is given by $\vec{x}^T L(G)\vec{x}/\vec{x}^T\vec{x}$.

Alon [Alo86] and Sinclair and Jerrum [SJ88] proved that graphs with small Fiedler value have a good ratio cut (Alon's theorem actually demonstrates the existence of a small vertex separator). A corollary of an extension of their work by Mihail [Mih89] demonstrates that one can obtain a good ratio cut from any vector with small Rayleigh quotient that is perpendicular to the all-ones's vector (although this is not explicitly stated in her work). In the full version of this paper, we present a new proof of Mihail's theorem (see also [AM85, Fil91, DS91, Mih89] and [Moh89] for a tighter bound).

Theorem 3 (Mihail). *Let $G = (V,E)$ be a graph on n nodes of maximum degree Δ, let $L(G)$ be its Laplacian matrix, and let ϕ be its isoperimetric number. For any vector $\vec{x} \in R^n$ such that $\sum_{i=1}^n x_i = 0$,*

$$\frac{\vec{x}^T L(G)\vec{x}}{\vec{x}^T\vec{x}} \geq \frac{\phi^2}{2\Delta}.$$

Moreover, there is an s so that the cut

$$(\{i : x_i \leq s\}, \{i : x_i > s\})$$

has ratio at most $\phi^2/(2\Delta)$.

2.3. Spectral Partitioning Methods

Let $\vec{u} = (u_1, \ldots, u_n)$ be a Fiedler vector of the Laplacian of a graph G. The idea of spectral partitioning is to find a *splitting value* s with which to partition the vertices of G into the set such that $u_i > s$ and the set such that $u_i \leq s$. We call such a partition a *Fiedler cut*. Popular choices for the splitting value s are: *bisection cut*, in which s is the median of $\{u_1, \ldots, u_n\}$; *ratio cut*, in which s is the value that gives the best cut ratio; *sign cut*, in which s is equal to 0; and *gap cut*, in which s is a value in the largest gap in the sorted list of Fiedler vector components. Other variations have been proposed.

In this paper, we will analyze the spectral method that uses the splitting value that achieves the best cut ratio. We will show that, for bounded-degree planar graphs and well-shaped meshes, it always finds a good ratio cut. Because of Theorem 3, an approximation to a Fiedler vector will suffice.

Guattery and Miller [GM95] have shown that there exist bounded-degree planar graphs on n vertices with constant-size separators for which spectral bisection and spectral sign cuts give separators that cut $n/3$ edges. Our bound on the Fiedler values of bounded degree planar graphs implies that they have Fiedler cuts of ratio $O(1/\sqrt{n})$. By Lemma 1, our result implies that a bisector of size $O(\sqrt{n})$ can be found by repeatly finding Fiedler cuts. Such repetition is necessary. In the full paper, we extend the results of Guattery and Miller to show that this repeated application of Fiedler cuts is required, even for some quite natural graphs. We will show that, for any constant δ in the range $0 < \delta \leq 1/2$, there are natural families of well-shaped two-dimensional meshes that have no Fiedler cut of small ratio that is also a δ-separator. We discuss why these graphs exist as well as why they might fail to arise in practice.

3. The Fiedler value of planar graphs

In this section, we will prove that the Fiedler value of every bounded-degree planar graph is $O(1/n)$. Our proof establishes and exploits a connection between the Fiedler value and geometric embeddings of graphs. We obtain the eigenvalue bound by demonstrating that every planar graph has a "nice" embedding in Euclidean space.

A bound of $O(1/\sqrt{n})$ can be placed on the Fiedler value of any planar graph by combining the planar separator theorem of Lipton and Tarjan [LT79] with the fact that $\lambda_2/2 \leq \phi(G)$. Bounds of $O(1/n)$ on the Fiedler values of planar graphs were previously known for graphs such as regular grids [PSL90], quasi-uniform graphs [GK95], and bounded-degree trees. Bounds on the Fiedler values of regular grids and quasi-uniform graphs essentially follow from the fact that the diameters of these graphs are large (see [Chu89]). Bounds on trees can be obtained from the fact that every bounded-degree tree has a δ-separator of size 1 for some constant δ in the range $0 < \delta < 1/2$ that depends only on the degree. However, in order to estimate the Fiedler value of general bounded-degree planar graphs and well-shaped meshes, we need different techniques.

We denote the standard l_2 norm of a vector $\vec{x}$ in Euclidean space by $\|\vec{x}\| = \sqrt{x^T x}$. We relate the quality of an embedding of a graph in Euclidean space with its Fiedler value by the following lemma:

Lemma 4 (embedding lemma). *Let $G = (V, E)$ be a graph. Then λ_2, the Fiedler value of G, is given by*

$$\lambda_2 = \min \frac{\sum_{(i,j)\in E} \|\vec{v}_i - \vec{v}_j\|^2}{\sum_{i=1}^n \|\vec{v}_i\|^2},$$

where the minimum is taken over vectors $\{\vec{v}_1, \ldots, \vec{v}_n\} \subset R^n$ such that $\sum_{i=1}^n \vec{v}_i = \vec{0}$, where $\vec{0}$ denotes the all-zeroes vector.

Proof: Consider the n^2 by n^2 matrix consisting of n diagonal blocks of $L(G)$. The eigenvalues of this matrix are the same as the eigenvalues of $L(G)$. The lemma follows because the condition that the sum of the vectors be $\vec{0}$ implies that the expression minimized above is just the Rayleigh quotient of a vector orthogonal to the eigenvectors with eigenvalue zero. $\square$

Our method of finding a good geometric embedding of a planar graph is similar to the way in which Miller, Teng, Thurston, and Vavasis [MTTV96a] directly find good separators of planar graphs.

We first find an embedding of the graph on the plane by using the "kissing disk" embedding of Koebe, Andreev, and Thurston [Koe36, And70a, And70b, Thu88]:

Theorem 5 (Koebe-Andreev-Thurston). *Let G be a planar graph with vertex set $V = \{1, \ldots, n\}$ and edge set E. Then, there exists a set of disks $\{D_1, \ldots, D_n\}$ in the plane with disjoint interiors such that D_i touches D_j if and only if $(i, j) \in E$.*

Such an embedding is called a *kissing disk* embedding of G.

The analogue of a disk on the sphere is a *cap*. A *cap* is given by the intersection of a half-space with the sphere, and its boundary is a circle. We define *kissing caps* analogously with kissing disks. Following [MTTV96a], we use stereographic projection to map the kissing disk embedding of a graph on the plane to a kissing cap embedding on the sphere (See Section 4 for more information on stereographic projection). In Theorem 9, we will show that we can find a sphere preserving map that sends the *centroid* (also known as the *center of gravity* or *center of mass*) of the centers of the caps to the center of the sphere. Using this theorem, we can bound the eigenvalues of planar graphs:

Theorem 6. *Let G be a planar graph on n nodes of degree at most Δ. Then, the Fiedler value of G is at most*

$$\frac{8\Delta}{n}.$$

Accordingly, G has a Fiedler cut of ratio $O(1/\sqrt{n})$, and one can iterate Fiedler cuts to find a bisector of size $O(\sqrt{n})$.

Proof: By Theorem 5 and Theorem 9, there is a representation of G by kissing caps on the unit sphere so that the centroid of the centers of the caps is the center of the sphere. Let $\vec{v}_1, \ldots, \vec{v}_n$ be the centers of these caps. Make the center of the sphere the origin, so that $\sum_{i=1}^n \vec{v}_i = \vec{0}$.

Let $r_1, \ldots, r_n$ be the radii of the caps. If cap i kisses cap j, then the edge from $\vec{v}_i$ to $\vec{v}_j$ will have length at most $(r_i + r_j)^2$. As this is at most $2(r_i^2 + r_j^2)$, we can divide the contribution of this edge between the two caps. That is, we write

$$\sum_{(i,j)\in E} \|\vec{v}_i - \vec{v}_j\|^2 \leq 2\Delta \sum_{i=1}^n r_i^2.$$

But, because the caps do not overlap,

$$\sum_{i=1}^n \pi r_i^2 \leq 4\pi.$$

Moreover, $\|\vec{v}_i\| = 1$ because the vectors are on the unit sphere.

Applying the embedding lemma, we find that the Fiedler value of G is at most

$$\frac{\sum_{(i,j)\in E} \|\vec{v}_i - \vec{v}_j\|^2}{\sum_{i=1}^{n} \|\vec{v_i}\|^2} \leq \frac{8\Delta}{n}.$$

Given the bound on the Fiedler value, the ratio achievable by a Fiedler cut follows from Theorem 3 and the corresponding bisector size follows from Lemma 1. $\square$

4. Sphere-preserving maps

Let B^d be the unit ball in d dimensions:

$$\left\{ (x_1, \ldots, x_d) | \sum_{i=1}^{n} x_i^2 \leq 1 \right\}.$$

Let S^d denote the sphere defining the surface of B^d. A *sphere-preserving* map from S^d to S^d is a continuous function that sends every sphere (of lower dimension) contained in S^d to a sphere in S^d and such that every sphere in S^d has a pre-image under the map that is also a sphere. Familiar sphere-preserving maps include rotations and the map that sends each point to its antipode.

We will make use of a slightly larger family of sphere-preserving maps. We obtain this family by first considering sphere-preserving maps between the sphere and the plane. Let H^d be the hyperplane tangent to S^d at $(-1, 0, \ldots, 0)$. One can map H^d to S^d by *stereographic projection*:

$$\begin{aligned} &\Pi : H^d \to S^d \quad \text{by} \\ &\quad \Pi(z) = \text{the intersection of } S^d \text{ with} \\ &\quad \text{the line connecting } z \text{to } (1, 0, \ldots, 0). \end{aligned}$$

Similarly, one defines a map $\Pi^{-1} : S^d \to H^d$ that sends a point $z \in S^d$ to the intersection of H^d with the line through z and $(1, 0, \ldots, 0)$. Note that Π^{-1} is not well-defined at $(1, 0, \ldots, 0)$. To fix this, we add the point ∞ to the hyperplane H^d, and define $\Pi^{-1}(1, 0, \ldots, 0) = \infty$ as well as $\Pi(\infty) = (1, 0, \ldots, 0)$.

For any point $\alpha \in S^d$, we define Π_α to be the stereographic projection from the plane perpendicular to S^d at α, and we let Π_α^{-1} be its inverse (so, $\Pi(\infty) = -\alpha$). One can show that the maps Π_α and Π_α^{-1} are sphere-preserving maps (see [HCV52] or [MTTV96a] for a proof).

Sphere-preserving maps in the plane include rigid motions of the plane as well as dilations (and other mobius transformations). We will obtain sphere-preserving maps in the sphere by applying a projection onto a plane, then applying a dilation of the plane, and then mapping back by stereographic projection. For $\alpha \in S^d$ and $a \geq 0$, we define D_α^a to be the map that dilates the hyperplane perpendicular to S^d at α by a factor of a (note that $D_\alpha^a(\infty) = \infty$). For example,

$$D^a_{(-1,0,\ldots,0)} : (-1, x_2, \ldots, x_d) \mapsto (-1, ax_2, \ldots, ax_d).$$

As the composition of sphere-preserving maps is again a sphere-preserving map, we can now define the sphere-preserving maps that we will use. For any α such that $\|\alpha\| < 1$, define $f_\alpha(z)$ by

$$f_\alpha(z) = \Pi_{\alpha/\|\alpha\|} \left(D^{1-\|\alpha\|}_{\alpha/\|\alpha\|} (\Pi^{-1}_{\alpha/\|\alpha\|}(z)) \right).$$

It is routine to verify that f_α is continuous. We wish to extend the definition of f_α to α on S^2, even though the resulting maps will not be continuous. For $\|\alpha\| = 1$, we define

$$f_\alpha(z) = \begin{cases} -\alpha & \text{if } z = -\alpha, \text{ and} \\ \alpha & \text{otherwise.} \end{cases}$$

We will now examine the effect of the maps f_α on arrangements of spherical caps on S^d. Recall that a spherical cap on S^d is a connected region of S^d whose boundary is a $(d-1)$-dimensional sphere. Thus, the image of a cap under a map f_α is determined by the image of its boundary along with a point in its interior. For a cap C on S^d, let $p(C)$ denote the point on S^d that is the center of C (i.e., the point inside C that is equidistant from its boundary). We want to show that, for any arrangement of caps $\{C_1, \ldots, C_n\}$ on S^d, there is an $\alpha \in S^d$ so that the centroid of $\{p(f_\alpha(C_1)), \ldots, p(f_\alpha(C_n))\}$ is the origin. But first, we must exclude some degenerate cases:

Definition 7. An arrangement of caps $\{C_1, \ldots, C_n\}$ in S^d is *well-behaved* if there is no point that belongs to at least half of the caps.

Remark 8. *All of the arrangements of caps obtained from graphs contained in the other sections of this paper are well-behaved. Otherwise, the induced graphs would have cliques on half of their vertices and no small separators.*

Theorem 9. *For any well-behaved arrangement of caps $\{C_1, \ldots, C_n\}$ in S^d, there is a sphere-preserving map f_α such that the centroid of the centers of $\{f_\alpha(C_1), \ldots, f_\alpha(C_n)\}$ is the origin.*

Proof: We will show that there is an α so that $\|\alpha\| < 1$ and

$$\frac{\sum_{i=1}^{n} p(f_\alpha(C_i))}{n} = \vec{0}.$$

Consider the map from α to the centroid of

$$\{p(f_\alpha(C_1)), \ldots, p(f_\alpha(C_n))\}.$$

We want to show that $\vec{0}$ has a preimage under this map. This would be easier if the map were continuous, but it is not continuous for $\|\alpha\| = 1$: as $-\alpha$ crosses the boundary of C_i, $p(f_\alpha(C_i))$ jumps from one side of the sphere to the other.

To fix this problem, we construct a slightly modified map that is continuous. Because the set of caps is well-behaved, we can choose an $\epsilon > 0$ so that, for all α such that $\|\alpha\| \geq 1 - \epsilon$, most of the caps $\{f_\alpha(C_1), \ldots, f_\alpha(C_n)\}$ are entirely contained within the ball of radius $1/2n$ around $\alpha/\|\alpha\|$. This implies that f_α does not map the centroid of the centers of the caps to the origin. For $\alpha \in B^d$, we now define the map

$$\phi(\alpha) = \frac{\sum_{i=1}^n w(C_i, \alpha) f_\alpha(p(C_i))}{n},$$

where the weight function w is given by

$$w(C, \alpha) = \begin{cases} (2 - d(\alpha, C))/\epsilon & \text{if } d(\alpha, C) \geq 2 - \epsilon, \text{ and} \\ 1 & \text{otherwise,} \end{cases}$$

where by $d(\alpha, C)$, we mean the greatest distance from α to a point in the cap C (for example, if $-\alpha/\|\alpha\| \in C$, then $d(\alpha, C) = 1 + \|\alpha\|$). We have chosen w to be a continuous function of α that goes to zero as $-\alpha$ approaches the boundary of a cap; so, $\phi(\alpha)$ is also a continuous function.

From the fact that $\{C_1, \ldots, C_n\}$ is well-behaved, it is easy to verify that, for $\alpha \in S^d$, $\phi(\alpha)$ lies on the line connecting $\vec{0}$ to α and is closer to α than it is to $-\alpha$. Combined with Lemma 10, this implies that there is an α such that $\phi(\alpha) = \vec{0}$. Our choice of ϵ implies that $\|\alpha\| < 1 - \epsilon$; so, all of the $w(\alpha, C_i)$ terms are 1 and f_α is the map we are looking for. $\square$

Lemma 10. *Let $\phi : B^d \to B^d$ be a continuous function such that, for all $\alpha \in S^d$, $\phi(\alpha)$ lies on the line connecting α with $\vec{0}$ and is closer to α than it is to $-\alpha$. Then, there exists an $\alpha \in B^d$ such that $\phi(\alpha) = \vec{0}$.*

Proof: Assume, by way of contradiction, that there is no point $\alpha \in B^d$ such that $\phi(\alpha) = \vec{0}$. Now, consider the map $b(\phi(\alpha))$, where $b : B^d - \left\{\vec{0}\right\} \to S^d$ by $b(z) = z/\|z\|$. Since b is a continuous map, $b \circ \phi$ is a continuous map of B^d onto S^d that is the identity on S^d. Then $z \mapsto -b(\phi(z))$ is a map from B^d onto S^d that has no fixed point. This contradicts Brouwer's Fixed Point Theorem, which says that every continuous map from B^d into B^d has a fixed point. $\square$

We have shown that, for all well-behaved collections of balls in H^d, there is a sphere preserving map from H^d to S^d so that the centroid of the centers of the caps is the origin. We now show that one can find such a map by performing a rigid motion of H^d followed by a dilation of H^d followed by stereographic projection. We will need this stronger theorem when we bound the Fiedler values of well-shaped meshes.

Definition 11. An arrangement of balls $\{D_1, \ldots, D_n\}$ in H^d is *well-behaved* if there is no point that belongs to at least half of the balls.

Theorem 12. *Let $\{D_1, \ldots, D_n\}$ be a well-behaved collection of balls. Then, there is a point $x \in H^d$ and an $a > 0$ so that the sphere preserving map*

$$g_{x,a} : z \mapsto \Pi(a(z - x))$$

sends the balls to a collection of caps, the centroid of whose centers is the origin.

Proof: [**sketch**] For an $\alpha \in S^d$, consider the map $g_{\Pi^{-1}(\alpha),(1-\|\alpha\|)}$ followed by a rotation of the sphere that sends $(-1, 0, \ldots, 0)$ to α. As we did in the proof of Theorem 9, we can construct a continuous map from α to a weighted centroid of the centers of the caps, which for $\alpha \in S^d$ sends α to a point on the line segment between α and $\vec{0}$. We can then apply Lemma 10 to prove that there is some map α such that the map $g_{\Pi^{-1}(\alpha),(1-\|\alpha\|)}$ sends the centroid of the centers of the caps to the origin. $\square$

5. The Spectra of Well-Shaped Meshes

One of the main applications of the spectral method is the partitioning of meshes for parallel numerical simulations. Many experments demonstrate the effectiveness of this method [BS92, HL92, HL93, PSL90, Sim91, Wil90]. In this section, we explain why the spectral method finds such good partitions of well-shaped meshes.

5.1. Modeling Well-Shaped Meshes

The graphs that we consider are defined by neighborhood systems. A *neighborhood system* is a set of closed balls in Euclidean space. A *k-ply* neighborhood system is one in which no point is contained in the interior of more than k of the balls. Given a neighborhood system, $\Gamma = \{B_1, \ldots, B_n\}$, we define the *intersection graph* of Γ to be the undirected graph with vertex set $V = \{B_1, \ldots, B_n\}$ and edge set

$$E = \{(B_i, B_j) : B_i \cap B_j \neq \emptyset\}.$$

We will use *overlap* graphs to model well-shaped meshes (Miller *et al* [MTTV96a]). An overlap graph is based on a k-ply neighborhood system. The neighborhood system and a parameter, $\alpha \geq 1$, define an overlap graph: Let $\alpha \geq 1$, and let $\Gamma = \{B_1, \ldots, B_n\}$ be a k-ply neighborhood system in R^d. The *α-overlap graph* of Γ is the graph with vertex set $\{B_1, \ldots, B_n\}$ and edge set

$$\{(B_i, B_j) : (B_i \cap (\alpha \cdot B_j) \neq \emptyset) \text{ and } ((\alpha \cdot B_i) \cap B_j \neq \emptyset)\},$$

where by $\alpha \cdot B$, we mean the ball whose center is the same as the center of B and whose radius is larger by a multiplicative factor of α.

Overlap graphs are good models for well-shaped meshes because each well-shaped mesh in two, three, or higher dimensions is a *bounded-degree subgraph* of some overlap graph (for suitable choices of the parameters α and k) [MTTV96a, MTTW95, Ten96, MV91].

5.2. Spherical Embeddings of Overlap Graphs

In this section, we show that an α-overlap graph is a subgraph of the intersection graph obtained by projecting its neighborhoods onto the sphere and then dilating each by an $O(\alpha)$ factor. By choosing the proper projection, we are able to use this fact to bound the eigenvalues of these graphs.

In this section, we use the following notation: Capital letters denote balls in R^d. If A is a ball in R^d, then we will use A' to denote its image on the sphere S^{d+1} under stereographic projection. If α is positive and A is a ball of radius r, then $\alpha \cdot A$ is the ball with the same center as A and radius αr. Similarly, if A' is a spherical cap of spherical radius r, then $\alpha \cdot A'$ is the spherical cap with the same center as A' and radius αr. V_d is the volume of a unit d-dimensional ball and A_d is the surface volume of a unit d-dimensional ball.

Theorem 13. *Let $\alpha \geq 1$ and let A and B be balls in R^d such that*

$$(A \cap (\alpha \cdot B) \neq \emptyset) \text{ and } ((\alpha \cdot A) \cap B \neq \emptyset).$$

Then, $(\pi\alpha + \alpha + \pi) \cdot A'$ touches $(\pi\alpha + \alpha + \pi) \cdot B'$.

Our proof uses two lemmas that handle orthogonal special cases.

Lemma 14. *Let A and C be balls in R^d equidistant from the origin and having the same radius. Let A' and C' be their images under stereographic projection onto S^{d+1}. If $\alpha \cdot A$ touches $\alpha \cdot C$, then $(\alpha\pi/2) \cdot A'$ touches $(\alpha\pi/2) \cdot C'$.*

Lemma 15. *Let A and B be balls in R^d so that the center of A, the center of B, and the origin are colinear and the origin does not lie on the line segment between the center of A and the center of B. If A is closer to the origin than B and $\alpha \cdot A$ touches B, then $\alpha \cdot A'$ touches B'.*

Proof: [**of Theorem 13**] Let A and B be any two balls in R^d and let A' and B' be their images under stereographic projection on S^{d+1}. Assume that $\alpha \cdot A$ touches B and $\alpha \cdot B$ touches A. We will show that $(\pi\alpha + \alpha + \pi) \cdot A'$ touches $(\pi\alpha + \alpha + \pi) \cdot B'$.

Assume, without loss of generality, that A is at least as large as B. Let C be the disk of the same distance to the origin as A and congruent to A that is closest to B. Then, the centers of C and B are colinear with the origin. Let C' be the image of C. Since C is closer to B than A is, $\alpha \cdot C$ touches B and $\alpha \cdot B$ touches A. By Lemma 15, $\alpha \cdot C'$ touches $\alpha \cdot B'$.

The distance between the centers of A and B is less than $(\alpha + 1)$ times the radius of A (because we assume that A is at least as large as B). The same holds for the distance between the center of C and the center of B. Therefore, $(\alpha + 1) \cdot A$ touches $(\alpha + 1) \cdot C$, so Lemma 14 implies that $\pi(\alpha+1)/2 \cdot A'$ touches $\pi(\alpha+1)/2 \cdot C'$. Since A' and C' have the same spherical radius, $\alpha \cdot C' \subset (\pi(\alpha+1)+\alpha)A'$. Thus, $(\pi\alpha+\alpha+\pi) \cdot A'$ must touch $(\pi\alpha+\alpha+\pi) \cdot B'$. □

5.3. The Spectral Bound

We now show that the Fiedler value of a bounded degree subgraph of an α-overlap graph is small.

Theorem 16. *If G is a subgraph of an α-overlap graph of a k-ply neighborhood system in R^d and the maximum degree of G is Δ, then the Fiedler value of $L(G)$ is bounded by $\gamma_d \Delta \alpha^2 (k/n)^{2/d}$, where $\gamma_d = 2(\pi + 1 + \pi/\alpha)^2 (A_{d+1}/V_d)^{2/d}$. Accordingly, G has a Fiedler cut of ratio $O\big(\Delta\alpha(k/n)^{1/d}\big)$, and one can iterate Fiedler cuts to obtain a bisector of size $O\big(\Delta\alpha k^{1/d} n^{1-1/d}\big)$.*

Proof: Let $\Gamma = \{B_1, ..., B_n\}$ be the k-ply neighborhood system whose intersection graph contains G. By Theorem 12, there is a stereographic projection Π from R^d onto a particular sphere S^{d+1} so that the centroid of the centers of the images of the neighborhoods is the center of the sphere.

Let $\Pi(\Gamma) = \{B'_i, ..., B'_n\}$ be the images of the balls in Γ under Π. Let r_i be the radius of B'_i. Because $V_d r^d \leq volume(B'_i)$, We know that

$$\sum_{i=1}^{n} V_d r_i^d \leq \sum_{i=1}^{n} volume(B'_i) \leq kA_{d+1}.$$

By Theorem 13, G is a subgraph of the intersection graph of $\{(\pi\alpha + \alpha + \pi) \cdot B'_i : 1 \leq i \leq n\}$. Thus, by Lemma 4,

$$\begin{aligned} \lambda_2(L(G)) &\leq \frac{\sum_{i=1}^{n} 2\Delta(\pi\alpha + \alpha + \pi)^2 r_i^2}{n} \\ &\leq (2\Delta)(\pi\alpha + \alpha + \pi)^2 \left(\frac{A_{d+1}}{V_d}\right)^{2/d} \left(\frac{k}{n}\right)^{2/d}. \end{aligned}$$

Given the bound on the Fiedler value, the ratio achievable by a Fiedler cut follows from Theorem 3 and the corresponding bisector size follows Lemma 1. □

6. Acknowledgements

We thank Michael Mandell for advice on algebraic topology, John Gilbert, Nabil Kahale, Gary Miller, and Horst Simon for helpful discussions on the spectra of graphs, and Edmond Chow, Alan Edelman, Steve Guattery, Bruce Hendrickson, Satish Rao, Ed Rothberg, Dafna Talmor, and Steve Vavasis for comments on an early draft of this paper.

References

[AK95] C. J. Alpert and A. B. Kahng. Recent directions in netlist partitioning: A survey. *Integration: The VLSI Journal*, 19:1–81, 1995.

[Alo86] N. Alon. Eigenvalues and expanders. *Combinatorica*, 6(2):83–96, 1986.

[AM85] N. Alon and V. Milman. λ_1, isoperimetric inequalities for graphs, and superconcentrators. *J. Comb. Theory Series B*, 38:73–88, 1985.

[And70a] E. M. Andreev. On convex polyhedra in Lobacevskii space. *Math. USSR Sbornik*, 10(3):413–440, 1970.

[And70b] E. M. Andreev. On convex polyhedra of finite volume in Lobacevskii space. *Math. USSR Sbornik*, 12(2):270–259, 1970.

[AST90] N. Alon, P. Seymour, and R. Thomas. A separator theorem for graphs with an excluded minor and its applications. In *Proceedings of the 22nd Annual ACM Symposium on Theory of Computing*, pages 293–299. ACM, 1990.

[Bar82] E. R. Barnes. An algorithm for partitioning the nodes of a graph. *SIAM J. on Algebraic and Discrete Methods*, 4(3):541–550, 1982.

[BH84] E. R. Barnes and A. J. Hoffman. Partitioning, spectra and linear programming. *Progress in Combinatorial Optimization*, pages 13–25, 1984.

[Bop87] R. Boppana. Eigenvalues and graph bisection: An average-case analysis. In *28th Annual Symposium on Foundations of Computer Science*, pages 280–285, Los Angeles, October 1987. IEEE.

[BS92] S. T. Barnard and H. D. Simon. A fast multilevel implementation of recursive spectral bisection for partitioning unstructured problems. Technical Report RNR–92–033, NASA Ames Research Center, 1992.

[Che70] J. Cheeger. A lower bound for the smallest eigenvalue of the Laplacian. In R. C. Gunning, editor, *Problems in Analysis*, pages 195–199. Princeton University Press, 1970.

[Chu89] F. Chung. Diamaters and eigenvalues. *Journal of the American Mathematical Society*, 2(2):187–196, 1989.

[CR87] T. F. Chan and D. C. Resasco. A framework for the analysis and construction of domain decomposition preconditioners. Technical Report CAM-87-09, UCLA, 1987.

[CS93] T. F. Chan and B. Smith. Domain decomposition and multigrid algorithms for elliptic problems on unstructured meshes. *Contemporary Mathematics*, pages 1–14, 1993.

[CSZ93] P. K. Chan, M. Schlag, and J. Zien. Spectral k-way ratio cut partitioning and clustering. In *Symp. on Integrated Systems*, 1993.

[DG92] H. N. Djidjev and J. R. Gilbert. Separators in graphs with negative and multiple vertex weights. Technical Report CS-92-07, Xerox PARC, Palo Alto, 1992.

[DH72] W. E. Donath and A. J. Hoffman. Algorithms for partitioning of graphs and computer logic based on eigenvectors of connection matrices. *IBM Technical Disclosure Bulletin*, 15:938–944, 1972.

[DH73] W. E. Donath and A. J. Hoffman. Lower bounds for the partitioning of graphs. *J. Res. Develop.*, 17:420–425, 1973.

[DS91] P. Diaconis and D. Strook. Geometric bounds for eigenvalues of Markov chains. *The Annals of Applied Probability*, 1(1):36–61, 1991.

[Fie73] M. Fiedler. Algebraic connectibity of graphs. *Czechoslovak Mathematical Journal*, 23(98):298–305, 1973.

[Fie75a] M. Fiedler. Eigenvectors of acyclic matices. *Czechoslovak Mathematical Journal*, 25(100):607–618, 1975.

[Fie75b] M. Fiedler. A property of eigenvectors of nonnegative symmetric matrices and its applications to graph theory. *Czechoslovak Mathematical Journal*, 25(100):619–633, 1975.

[Fil91] J. A. Fill. Eigenvalue bounds on convergence to stationarity for nonreversible Markov chains, with an application to the exclusion process. *The Annals of Applied Probability*, 1(1):62–87, 1991.

[GHT84] J. Gilbert, J. Hutchinson, and R. Tarjan. A separation theorem for graphs of bounded genus. *Journal of Algorithms*, 5:391–407, 1984.

[Gil80] J. R. Gilbert. *Graph Separator Theorems and Sparse Gaussian Elimination.* PhD thesis, Computer Science Department, Stanford University, 1980.

[GK95] J. R. Gilbert and N. Kahale. personal communication, 1995.

[GM95] S. Guattery and G. L. Miller. On the performance of the spectral graph partitioning methods. In *Proceedings of The Second Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 233–242. ACM-SIAM, 1995.

[HCV52] D. Hilbert and S. Cohn-Vossen. *Geometry and the Imagination.* Chelsea Publishing Company, New York, 1952.

[HK92] L. Hagen and A. B. Kahng. New spectral methods for ratio cut partitioning and clustering. *IEEE Transactions on Computer-Aided Design*, 11(9):1074–1085, September 1992.

[HL92] B. Hendrickson and R. Leland. An impoved spectral graph partitioning algorithm for mapping parallel computation. Technical Report SAND92-1460, Sandia National Lab., December 1992.

[HL93] B. Hendrickson and R. Leland. The Chaco user's guide, Version 1.0. Technical Report SAND93-2339, Sandia National Lab., 1993.

[Koe36] P. Koebe. Kontaktprobleme der konformen Abbildung. *Ber. Verh. Sächs. Akademie der Wissenschaften Leipzig, Math.-Phys. Klasse*, 88:141–164, 1936.

[LR88] F. T. Leighton and S. Rao. An approximate max-flow min-cut theorem for uniform multicommodity flow problems with applications to approximation algorithms. In *29th Annual Symposium on Foundations of Computer Science*, pages 422–431, 1988.

[LT79] R. J. Lipton and R. E. Tarjan. A separator theorem for planar graphs. *SIAM J. of Appl. Math.*, 36:177–189, April 1979.

[Mih89] M. Mihail. Conductance and convergence of Markov chains – a combinatorial treatment of expanders. In *30th Annual Symposium on Foundations of Computer Science*, pages 526–531. IEEE, 1989.

[Moh89] B. Mohar. Isoperimetric numbers of graphs. *Journal of Combinatorial Theory, Series B*, 47:274–291, 1989.

[MT90] G. L. Miller and W. Thurston. Separators in two and three dimensions. In *Proceedings of the 22nd Annual ACM Symposium on Theory of Computing*, pages 300–309, Baltimore, May 1990. ACM.

[MTTV96a] G. L. Miller, S.-H. Teng, W. Thurston, and S. A. Vavasis. Finite element meshes and geometric separators. *SIAM J. Scientific Computing*, page to appear, 1996.

[MTTV96b] G. L. Miller, S.-H. Teng, W. Thurston, and S. A. Vavasis. Separators for sphere-packings and nearest neighborhood graphs. submitted to *J. ACM*, 1996.

[MTTW95] G. L. Miller, D. Talmor, S.-H. Teng, and N. Walkington. A Delaunay based numerical method for three dimensions: generation, formulation, and partition. In *Proceedings of the 27th Annual ACM Symposium on Theory of Computing*, pages 683–692, Las Vegas, May 1995. ACM.

[MTV91] G. L. Miller, S.-H. Teng, and S. A. Vavasis. An unified geometric approach to graph separators. In *32nd Annual Symposium on Foundations of Computer Science*, pages 538–547. IEEE, 1991.

[MV91] G. L. Miller and S. A. Vavasis. Density graphs and separators. In *Proceedings of the ACM-SIAM Symposium on Discrete Algorithms*, pages 331–336, 1991.

[PRS94] S. Plotkin, S. Rao, and W. D. Smith. Shallow excluded minors and improved graph decomposition. In *5th Symp. Discrete Algorithms*, pages 462–470. SIAM, 1994.

[PSL90] A. Pothen, H. D. Simon, and K.-P. Liou. Partitioning sparse matrices with eigenvectors of graphs. *SIAM J. Matrix Anal. Appl.*, 11:430–452, 1990.

[PSS82] B. Parlett, H. Simon, and L. Stringer. Estimating the largest eigenvalue with the Lanczos algorithm. *Mathematics of Computation*, 38:153–165, 1982.

[PSW92] A. Pothen, H. D. Simon, and L. Wang. Spectral nested dissection. Technical Report CS–92–01, Pennsylvania State University Department of Computer Science, 1992.

[Sim91] H. D. Simon. Partitioning of unstructured problems for parallel processing. *Computing Systems in Engineering*, 2(2/3):135–148, 1991.

[SJ88] A. J. Sinclair and M. R. Jerrum. Conductance and the rapid mixing property for markov chains: the approximation of the permanent resolved. In *Proceedings of the 20th Annual ACM Symposium on Theory of Computing*, pages 235–244. ACM, 1988.

[SJ89] A. Sinclair and M. Jerrum. Approximative counting, uniform generation and rapidly mixing Markov chains. *Information and Computation*, 82(1):93–133, July 1989.

[Ten91] S.-H. Teng. *Points, Spheres, and Separators: a unified geometric approach to graph partitioning.* PhD thesis, Carnegie-Mellon University, School of Computer Science, 1991. CMU-CS-91-184.

[Ten96] S.-H. Teng. Provably good partitioning and load balancing algorithms for parallel adaptive n-body simulation. *SIAM J. Scientific Computing (to appear)*, 1996.

[Thu88] W. P. Thurston. *The geometry and topology of 3-manifolds.* Princeton University Notes, 1988.

[Ung51] P. Ungar. A theorem on planar graphs. *Journal London Math Soc.*, 26:256–262, 1951.

[Wil90] R. D. Williams. Performance of dynamic load balancing algorithms for unstructured mesh calculations. Technical report, California Institute of Technology, 1990.