The Random Walk Construction of Uniform Spanning Trees and Uniform Labelled Trees

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Abstract

A random walk on a finite graph can be used to construct a uniform random spanning tree. We show how random walk techniques can be applied to the study of several properties of the uniform random spanning tree: the proportion of leaves, the distribution of degrees, and the diameter.

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1 Introduction

Let G be a finite connected graph on n vertices. A *random walk* on G is the discrete-time Markov chain with transition matrix P of the form

$$\begin{aligned} P(v,w) &= 1/r_v \quad \text{if } (v,w) \text{ is an edge} \\ &= 0 \qquad\ \ \text{if not} \end{aligned}$$

where r_v is the degree of v.

Let $(X_j; j \geq 0)$ be a random walk on G with X_0 arbitrary. For each vertex v let T_v be the first hitting time:

$$T_v = \min\{j \geq 0 : X_j = v\}.$$

It is elementary that the *cover time*

$$C = \max_v \ T_v$$

is finite (with probability 1). In terms of the first C steps of the walk we can define a subgraph $\mathcal{T}$ of G to consist of the $n-1$ edges

$$(X_{T_v-1}, X_{T_v}); v \neq X_0. \tag{1}$$

It is clear that $\mathcal{T}$ is a random spanning tree. It is not immediately clear that it is a *uniform* random spanning tree, but such is in fact the case.

Proposition 1 *Let $N(G)$ be the number of spanning trees t of G. Then $P(\mathcal{T} = t) = 1/N(G)$ for each t.*

One could regard $\mathcal{T}$ as a *rooted* tree with root X_0. Then by taking X_0 uniform on G, we make $\mathcal{T}$ uniform on the set of all rooted spanning trees of G.

The proof, given in section 2, is part of a known circle of ideas relating spanning trees to Markov chain stationary distributions. Clearly Proposition 1 yields an algorithm for generating a uniform spanning tree in time $O(EC)$. It is known [4] that $EC \leq n^3$, and [13] that for regular graphs $EC \leq 8n^2$. Andrei Broder [7] has independently noted Proposition 1 and discusses these algorithmic aspects at greater length. Our emphasis is rather on theoretical properties of uniform spanning trees, a topic on which there seems no literature at all. Proposition 1 opens the door to their study via modern random walk techniques. Proposition 5 bounds the probabilities of specified vertices being leaves of $\mathcal{T}$. Proposition 9 gives the degree of vertices in $\mathcal{T}$, where G has strong symmetry properties, and Proposition 10 gives asymptotics in that setting. Theorem 15 bounds the diameter of $\mathcal{T}$ on highly-connected graphs.

Of course, $N(G)$ is in principle calculable via the classical matrix-tree theorem given in many textbooks on graph theory. So one could in principle calculate $P(v \text{ is a leaf of } \mathcal{T})$ by calculating $N(G)$ and $N(G')$, where G' is the subgraph obtained by deleting v. But such direct combinatorial methods do not seem helpful for studying the diameter of $\mathcal{T}$, say, and apparently have not been studied even in the simple context of leaf probabilities.

Now let K_n denote the complete graph on vertices $1, \ldots, n$. If $\mathcal{T}_n$ is a uniform spanning tree of K_n then clearly $\mathcal{T}_n$ is precisely the uniform random tree on n labelled vertices. Thus Proposition 1, applied to the complete graph, yields a construction of this random tree; in section 2 we observe that the construction can be rephrased as the following two-stage procedure.

Algorithm 2 *Fix* $n \geq 2$.
(i) For $2 \leq i \leq n$ *connect vertex* i *to vertex* $V_i = \min(U_i, i-1)$*, where* $U_2, \ldots, U_n$ *are independent and uniform on* $1, \ldots, n$.
(ii) Relabel vertices $1, \ldots, n$ *as* $\pi(1), \ldots, \pi(n)$*, where* π *is a uniform random permutation of* $1, \ldots, n$.

Proposition 3 *The random tree* $\mathcal{T}_n$ *produced by Algorithm 2 is uniform:* $P(\mathcal{T}_n = t) = 1/n^{n-2}$ *for each tree* t *on vertices* $1, \ldots, n$.

There are several schemes for enumerating labelled trees (e.g. Prufer code), and any such scheme can be used to generate the uniform random labelled tree. Andrei Broder (personal communication) has observed that the enumeration scheme in [14] 2.3.4.4 Exercise 18 leads to a generation method which seems qualitatively similar, but not identical, to Algorithm 2. However, this algorithm does not seem to have been explicitly noted before. Its special feature is that when one considers the natural way of drawing the generated tree, one can easily intuit the limiting ($n \to \infty$) large-scale (diameter $\Theta(n^{1/2})$) behavior of uniform random trees, and define a limiting "continuum random tree" which determines all such limit properties. The topic is pursued in detail in [3].

One could seek to specialize Proposition 1 to other graphs, to produce specific efficient algorithms for generating uniform spanning trees on that specific graph. Algorithm 4 in section 2 does this for the complete bipartite graph. It is not clear whether this algorithm corresponds to some explicit simple enumeration of the spanning trees of the complete bipartite graph.

Finally, let us mention some fascinating work-in-progress by Robin Pemantle [18], who uses the random walk construction to discuss uniform random spanning trees of the *infinite* integer lattice Z^d.

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2 Proofs of Propositions 1 and 3

A frequently-rediscovered result (related to the matrix-tree theorem) gives the stationary distribution of a finite irreducible Markov chain. For each rooted tree t directed toward the root, let $q(t) = \prod P(v,w)$, the product taken over directed edges (v,w) of t. Let $p(v) = \sum_{t:\text{ root }(t)=v} q(t)$. Then the stationary distribution is proportional to $p(\cdot)$. This result has a natural probabilistic interpretation, as follows. Run the stationary chain X_j from time $-\infty$ to time 0, and construct the random tree $\mathcal{T}$ with directed edges $(X_{L_w}, X_{L_w+1}), w \neq X_0$ where L_w is the time of the final visit to w. Then it can be verified that $P(\mathcal{T} = t)$ is proportional to $q(t)$ and so, summing over trees t rooted at v, $P(X_0 = v)$ is proportional to $p(v)$. The author heard this interpretation from Anantharum and Tsoucas [5] and from Persi Diaconis (personal communication), who attributes it to Peter Doyle: it appears in an undergraduate thesis of a student of Doyle's [20].

Proposition 1 is formally merely a specialization of these ideas, in which we use the time-reversibility of random walks on graphs to express the result in terms of first hitting times instead of last exit times. A direct proof is sufficiently straightforward that it seems worth giving. Recall the Proposition asserts that the tree $\mathcal{T}$ constructed via (1) is uniform over the set of all spanning trees.

Proof of Proposition 1. Suppose first that G is r-regular. Let $\mathcal{S}$ be the set of rooted spanning trees in G. We can consider the random walk as a stationary process $(X_j; -\infty < j < \infty)$ indexed by the integers. Write S_j for the tree constructed by $(X_j, X_{j+1}, \ldots)$. That is, the tree rooted at X_j and with edges

$$(X_{T_v^j-1}, X_{T_v^j}); v \neq X_j$$

where

$$T_v^j = \min\{k \geq j : X_k = v\}.$$

Then $(S_j; -\infty < j < \infty)$ is a stationary $\mathcal{S}$ -valued Markov chain which (see below) is irreducible. Consider the transition matrix Q for this chain in reversed time:

$$Q(t,t') = P(S_{-1} = t' | S_0 = t).$$

Then
(a) Given t, there are exactly r trees t' such that $Q(t,t') = 1/r$, and $Q(t,t') = 0$ for other t'.
(b) Given t', there are exactly r trees t such that $Q(t,t') = 1/r$, and $Q(t,t') = 0$ for other t.
Indeed, given $S_0 = t$ with $\mathrm{root}(t) = v$, say, then X_{-1} has chance $1/r$ to be each of the r neighbors of v; each possibility leads to S_{-1} being some tree t', and there are no other possibilities for S_{-1}. A similar argument gives (b). Now (a) and (b) say that Q is doubly-stochastic, so its stationary distribution (the distribution of S_0) is uniform on $\mathcal{S}$. This establishes the result for rooted trees; the unrooted case follows.

In the non-regular case, write $r(t)$ for the degree of the root of t. Then (a) and (b) become:
(a') Given t, there are exactly $r(t)$ trees t' such that $Q(t,t') = 1/r(t)$, and $Q(t,t') = 0$ for other t'.
(b') Given t', there are exactly $r(t')$ trees t such that $Q(t,t') = 1/r(t)$, and $Q(t,t') = 0$ for other t.
Thus for fixed t',

$$\sum_t r(t)Q(t,t') = r(t')$$

and it follows that $\mathcal{S}$ has stationary distribution proportional to $r(t)$.

So we have shown that when X_0 has the (stationary) distribution proportional to r_v, the tree $\mathcal{T}$ constructed by Proposition 1 has $P(\mathcal{T} = t)$ proportional to $r(t)$. Thus conditional on $X_0 = w$ (for arbitrary w), $\mathcal{T}$ is uniform on rooted spanning trees rooted at w; and if we consider $\mathcal{T}$ as an unrooted tree it is uniform on the set of all unrooted spanning trees of G. Also, when X_0 is uniform, then $\mathcal{T}$ is uniform on all rooted spanning trees.

Finally, we need to check that the chain $(\mathcal{S}_j)$ is irreducible. It is enough to show that, for any spanning tree t, there exists a finite path $(x_0, x_1, \ldots, x_c)$ which visits every vertex and is such that the tree constructed via (1) is t. But this is easy: consider e.g. the "depth-first search" of the tree t.

Proof of Proposition 3. Recall the Proposition asserts that the random tree $\mathcal{T}_n$ constructed by Algorithm 2 is uniform over the set of all trees on vertices $\{1, 2, \ldots, n\}$.

Let $Z_0, Z_1, \ldots$ be independent uniform on $1, \ldots, n$. Let $\pi_1, \pi_2, \ldots$ be the distinct states hit first, second, ... by the process $\mathbf{Z}$, and let $\xi_1, \xi_2, \ldots$ be the corresponding first hitting times. Precisely,

$$\begin{aligned} \xi_1 &= 0 \\ \pi_1 &= Z_0 \\ \xi_{j+1} &= \min\{m > \xi_j : Z_m \text{ not in } \{\pi_1, \ldots, \pi_j\}\} \\ \pi_{j+1} &= Z_{\xi_{j+1}}. \end{aligned}$$

Let $L_{j+1} = Z_{\xi_{j+1}-1}$. Now consider the tree in which π_{j+1} is connected to L_{j+1}, for each $j \geq 1$. First, I assert that this is the uniform random labelled tree on n vertices. For the construction is unaffected if $\mathbf{Z}$ is replaced by the subsequence $\mathbf{Z}'$ in which terms Z_i identical with their predecessor Z_{i-1} are deleted. But $\mathbf{Z}'$ is a random walk on the complete graph, and the construction above is the construction of Proposition 1 on the complete graph, so the tree is a uniform spanning tree in the complete graph, i.e. the uniform labelled tree.

Thus we need to argue that this construction is equivalent to that of Algorithm 2. Regard π as a random permutation of $1, \ldots, n$. Clearly π is the *uniform* random permutation. The construction of the tree above can be rephrased as a two-stage construction.
(i) Connect $(j+1)$ to $\pi^{-1}(L_{j+1}), j = 1, \ldots, n-1$.
(ii) Relabel $1, \ldots, n$ as $\pi_1, \ldots, \pi_n$.
Now condition on the entire permutation π and consider the operation of stage (i). I assert that, regardless of the value of π, the entire stage (i) operates in the same way as stage (i) of Algorithm 2; this will complete the proof. To prove the assertion, fix j and condition on the "past" process $(Z_m : m \leq \xi_j)$. With probability $1 - j/n$ we have $\xi_{j+1} = \xi_j + 1$, which implies that $L_{j+1} = Z_{\xi_j}$ and hence $\pi^{-1}(L_j + 1) = j$. Otherwise, $\xi_{j+1} = \xi_j + M + 1$ for some random $M \geq 1$. Conditionally on $\{M = m\}$, we have:

$$(Z_{\xi_j+1}, \ldots, Z_{\xi_j+m}) \text{ are independent uniform}$$

$$\text{on the previously-visited states} \{\pi_1, \ldots, \pi_j\}.$$

So in particular, $L_{j+1} = Z_{\xi_j+m}$ is uniform on the previously-visited states $\{\pi_1, \ldots, \pi_j\}$, and so $\pi^{-1}(L_{j+1})$ is uniform on $\{1, \ldots, j\}$. Combining these two facts, we see that

$$P(\pi^{-1}(L_{j+1}) = u | Z_1, \ldots, Z_{\xi_j}) = 1/n \text{ for } 1 \leq u \leq j-1$$
$$P(\pi^{-1}(L_{j+1}) = j | Z_1, \ldots, Z_{\xi_j}) = 1 - (j-1)/n.$$

This conditional probability is unaffected by the value of π, and agrees with stage (i) of Algorithm 2.

Here is a similar algorithm for the complete bipartite (n_1, n_2) graph. The proof is similar, and so is omitted.

Algorithm 4 *(i) Choose* $(1,1)$ *or* $(2,1)$ *as the root, with probabilities* $\frac{n_1}{n_1+n_2}$, $\frac{n_2}{n_1+n_2}$ *respectively.*
(ii) Suppose we have constructed a bipartite tree with vertices $(1,j),\ j \leq j_1;\ (2,j),\ j \leq j_2$, *and suppose (w.l.o.g.) that the last vertex added was* $(1, j_1)$. *Then do one of the following.*
(a) With probability $\frac{n_2-j_2}{n_2}$, *add vertex* $(2, j_2+1)$ *and connect it to* $(1, j_1)$.
(b) With probability $\frac{j_1 j_2 (n_2-j_2)}{n_2(n_1 n_2 - j_1 j_2)}$, *add vertex* $(2, j_2+1)$ *and connect it to* $(1, J)$, *where* J *is chosen uniformly on* $1, \ldots, j_1$.
(c) With probability $\frac{j_2(n_2-j_2)}{n_1 n_2 - j_1 j_2}$, *add vertex* $(1, j_1+1)$ *and connect it to* $(2, J)$, *where* J *is chosen uniformly on* $1, \ldots, j_2$.
(iii) Finally, relabel $(1,j), j \leq n_1$ *as* $(1, \pi(j)), j \leq n_1$, *where* π *is a uniform random permutation of* $1, \ldots, n_1$*; and similarly for* $(2,j), j \leq n_2$.

Remark. In the special case of the complete graph, the tree-process of the first paragraph of this section becomes the following Markov chain on the set of all rooted trees on vertices $\{1, \ldots, n\}$. At each step pick a vertex v uniformly. If v is not the old root, remove the edge from v to its parent, create a new edge from v to the old root, and make v the new root. So the stationary distribution of this chain is the uniform distribution on all n^{n-1} rooted trees. This fact (easy to verify directly) is used by Fowler [11] to rederive distributions of vertex depths in the uniform random labelled tree.

3 Leaves of spanning trees

The rest of the paper gives some applications of the random walk construction of Proposition 1. We work with regular graphs for simplicity.

Proposition 5 *Let* $\mathcal{T}$ *be the uniform random spanning tree in a* r*-regular graph* G, *for* $r \geq 3$.

(i) $P(v \text{ is a leaf of } \mathcal{T}) \leq \exp(-\frac{r-1}{2r})$ *for all* $v \in G$.
(ii) $\mathrm{ave}_{v \in G} P(v \text{ is a leaf of } \mathcal{T}) \geq \alpha(r)$, *where*

$$\alpha(r) = \sum_{j=2}^{r-1} r^{-1}(1-j/r)(1-\prod_{i=1}^{j-1}(1-r^{-1}(1-i/r))).$$

Thus we have an absolute upper bound on probabilities of particular vertices being leaves of the random spanning tree. We cannot have such a lower bound - removing v may split G into r components, in which case v has degree r in all spanning trees. Instead, (ii) gives a lower bound on average leaf probabilities. One can calculate $\alpha(3) = 2/81, \alpha(r) \uparrow e^{-1/2} - 1/2 \approx 0.106$ as $r \to \infty$. For references to deterministic extremal bounds (e.g. the maximum number of leaves, over all spanning trees of G) see [12].

Proof of Proposition 5. As in Proposition 1 let X_j be the random walk. Fix v and let A be the set of neighbors of v. Let ξ_i be the time of the i'th visit to A. Let $\mathcal{T}_i$ be the tree constructed by the random walk up to time ξ_i. Define D_i by: $1+D_i$ is the degree of v in $\mathcal{T}_i$, with the convention that $D_i = 0$ if v has not been visited before time ξ_i. Note that $D_1 = 0$ and $D_{i+1} = D_i$ or $D_i + 1$. Let $\Gamma(\xi_i)$ be the number of *distinct* elements of A visited through time ξ_i. I assert

$$P(D_{i+1} = D_i + 1 | D_1, \ldots, D_i, \Gamma(\xi_i), X_{\xi_i}) = r^{-1}(1 - \Gamma(\xi_i)/r).$$

For the right side gives the conditional probability of the event

$$X_{\xi_i+1} = v \text{ and } X_{\xi_i+2} \text{ is a previously unvisited vertex.}$$

And $D_{i+1} = D_i + 1$ iff this event occurs.

Since $\Gamma(\xi_i) \leq i$, we obtain

$$P(D_{i+1} = D_i + 1 | D_1, \ldots, D_i) \geq r^{-1}(1 - i/r). \tag{2}$$

Next, for $1 \leq j \leq r$,

$$\begin{aligned} P(D_j = 0) &= \prod_{i=1}^{j-1} P(D_{i+1} = 0 | D_i = 0) \\ &\leq \prod_{i=1}^{j-1} (1 - r^{-1}(1 - i/r)) \text{ using (2).} \end{aligned} \tag{3}$$

We can now argue (i), because

$$
\begin{aligned}
P(v \text{ is a leaf of } \mathcal{T}) &\leq P(D_r = 0) \\
&\leq \prod_{i=1}^{r-1} (1 - r^{-1}(1 - i/r)).
\end{aligned}
$$

The bound in (i) follows using $1 - y \leq e^{-y}$.

To argue (ii), write $M = \min(i : D_i = 1) \leq \infty$. Write $1 + D_\infty$ for the degree of v in $\mathcal{T}$. Then

$$
D_\infty = 1_{(M<\infty)} + \sum_{j=1}^{\infty} (D_{j+1} - D_j) 1_{(D_j > 0)}
$$

and so

$$
\begin{aligned}
ED_\infty &= P(M < \infty) + \sum_{j=1}^{\infty} E(D_{j+1} - D_j) 1_{(D_j > 0)} \\
&\geq P(M < \infty) + \sum_{j=1}^{r-1} r^{-1}(1 - j/r) P(D_j > 0) \text{ using (2).}
\end{aligned}
$$

Now $M < \infty$ is the event that v is not a leaf of $\mathcal{T}$, so using (3) we find

$$
P(v \text{ is a leaf of } \mathcal{T}) \geq 1 - ED_\infty + \alpha(r).
$$

In any tree on n vertices the average degree equals $2 - 2/n$. So $\text{ave}_v\, ED_\infty = 1 - 2/n$ and (ii) follows.

For later use (Corollary 12) let us indicate how the argument for (i) leads to bounds for the entire distribution of D_∞, rather than just $P(D_\infty = 0)$. Let $c(i, r)$ be constants,

$$
c(0, r) = 0, \quad c(i, r) \leq c(i + 1, r) \leq c(i, r) + 1.
$$

Corollary 6 *Let $\mathcal{T}$ be the uniform random spanning tree in a r-regular graph G, for $r \geq 3$. Let $1 + D$ be the degree of v in $\mathcal{T}$. Let $c(i, r)$ be as above. Let $\Gamma(\xi_i)$ be the number of* distinct *elements of A visited through the time of the i'th visit to A, the neighbor-set of v. Let D_i^* be independent $\{0, 1\}$-valued r.v.'s with $P(D_i^* = 1) = r^{-1}(1 - c(i, r)/r)$ and let $D^* = \sum_i D_i^*$. Then for each $u \geq 0$,*

$$
P(D \leq u) \leq P(D^* \leq u) + P(\Gamma(\xi_i) > c(i, r) \textit{ for some } i).
$$

Proof. Write $\mathcal{F}_i$ for the σ-field of events up to time ξ_i. Write $G_i = \{\Gamma(\xi_i) \le c(i,r)\}$. The argument for (2) gives

$$P(D_{i+1} - D_i = 1|\mathcal{F}_i) \ge r^{-1}(1 - c(i,r)/r) \text{ on } G_i.$$

There is now a natural construction of $\{0,1\}$-valued D_i^* such that

$$P(D_{i+1}^* = 1|\mathcal{F}_i, D_1^*, \ldots, D_i^*) = r^{-1}(1 - c(i,r)/r)$$

and

$$D_{i+1} - D_i \ge D_{i+1}^* \text{ on } G_i.$$

Writing $D^* = \sum_i D_i^*$, we see $D \ge D^*$ on $\bigcap G_i$, and the result follows.

4 Degrees in highly-symmetric graphs

For highly-symmetric graphs it is reasonable to hope we can compute explicitly the degree of a vertex in the uniform random spanning tree. Proposition 9 below deals with the very special class of graphs G which satisfy the following (very strong) hypothesis. Recall [6] that G is vertex-transitive if its automorphism group acts transitively on vertices.

Hypothesis 7 *(i) G is vertex-transitive, with degree $r \ge 3$.*
(ii) For each $v \in G$ and distinct neighbors w_1, w_2, w_3 of v, there exists a graph automorphism γ such that

$$\gamma(v) = v, \gamma(w_1) = w_1, \gamma(w_2) = w_3, \gamma(w_3) = w_2.$$

This class includes the cube graphs Q_d, for example. This class is perhaps related to the class of *distance regular* graphs discussed exhaustively in [9].

We need some preliminaries about the random walk on such a graph G. Fix a vertex v with neighbors $\{w_1, \ldots, w_r\} = A$, say. Distinguish w_1 and write $B = \{w_2, \ldots, w_r\}$ for the "other neighbors". We shall consider for a while the random walk started at w_1, and write P_{w_1} for probabilities associated with this walk. Recall T_x is the first hitting time on x; similarly let

$$T_B = \min\{j \ge 0 : X_j \in B\}$$

be the first time some other neighbor is hit, and let T^+ be the first time the walk returns to w_1. The purpose of symmetry hypothesis (ii) is to imply the following. Let $w' \in A$ and let $B' = A \backslash \{w'\}$. Then

$$P_{w'}(X(T_{B'}) = w) = \frac{1}{r-1}; w \in B' \tag{4}$$

$$P_{w'}(X(T_{B'}) = w, X(T_{B'} - 1) = v) \text{ is uniform in } w. \tag{5}$$

We shall use several parameters (a, b, c, ψ, θ) of the random walk which are equivalent, in the sense that any one of them determines the others. Define

$$a = P_{w_1}(T_{w_2} < T^+). \tag{6}$$

By the symmetry hypothesis, for any pair of vertices z, z' distance 2 apart, a is the chance that the random walk started at z hits z' before returning to z. Now consider the chance the walk hits some other neighbor before returning to w_1:

$$b = P_{w_1}(T_B < T^+).$$

By symmetry there is chance $1/(r-1)$ that $X_{T_B} = w_2$; and if this does not occur, there is chance $1/2$ that the walk will thereafter hit w_2 before w_1. Thus

$$a/b = 1/(r-1) \ + \frac{r-2}{2(r-1)}$$

which rearranges to

$$b = 2a(1 - r^{-1}).$$

Now consider the walk started at w_1 and run until it either hits B or returns to w_1. Exactly one of the four alternatives below must happen: their probabilities are noted.

$$\begin{array}{rcl} (i) & \text{prob} = r^{-2} & X_1 = v, X_2 = w_1 \\ (ii) & \text{prob} = r^{-1} - r^{-2} & X_1 = v, X_2 \in B \\ (iii) & \text{prob} = c, \text{ say} & T_B < \min(T^+, T_v) \\ (iv) & \text{prob} = 1 - r^{-1} - c & T^+ < \min(T_B, T_v). \end{array}$$

Clearly b is the chance of (ii) or (iii), so $b = r^{-1} - r^{-2} + c$, which rearranges to

$$c = (2a - r^{-1})(1 - r^{-1}).$$

Recall now the *craps principle*. If a chance experiment has a set of alternate outcomes, and is repeated until the first time some one of a specified subset of outcomes occurs ("the ultimate outcome"), then the relative probabilities of the possible ultimate outcomes are exactly their original relative probabilities. We apply this to calculate

$$\psi = P_{w_1}(T_B < T_v)$$

using the experiment: "start a random walk at w_1 and continue until it returns to w_1". We want the chance that (iii) occurs before (i) or (ii), and the craps principle says

$$\psi = c/(c + r^{-1})$$

which rearranges to

$$\psi = 1 - \frac{1}{2ar - 2a + r^{-1}}. \tag{7}$$

Similarly, consider

$$\theta = P_{w_1}(X_{T_B - 1} \neq v).$$

This is the chance that (iii) occurs before (ii), so

$$\theta = c/(c + r^{-1} - r^{-2})$$

which rearranges to

$$\theta = 1 - \frac{1}{2ar}. \tag{8}$$

We now specify a certain joint distribution for random variables $\mathbf{V} = (V_0, V_1, V_2)$ and use it to state the main result of this section.

Construction 8 *(i) $V_0 \geq 1$ and, for $1 \leq j \leq r-1$,*

$$P(V_0 \geq j+1 | V_0 \geq j) = \frac{\psi(r-j)/(r-1)}{1 - \psi + \psi(r-j)/(r-1)}.$$

(ii) Given V_0, $P(V_1 = 0) = V_0/r$ and $P(V_1 = 1) = 1 - V_0/r$.
(iii) Given V_0, V_1, V_2 has Binomial$(r - V_0 - V_1, 1 - \theta)$ distribution.

Proposition 9 *Let G be a graph satisfying Hypothesis 7. Let $1 + D$ be the degree of a specified vertex v in the uniform random spanning tree $\mathcal{T}$ of G. Then D has the distribution of $V_1 + V_2$, for $\mathbf{V}$ as in Construction 8.*

Remarks. An explicit expression for this distribution can be found from the construction, but is not very appealing. One simple feature is

$$P(D = r-1) = (1 - 1/r)(2ar - 2a - 1/r)^{-1}(2ar)^{2-r}.$$

Note that the distribution depends only on (r, a) and not on the size n of the graph. The asymptotics are simpler: see Proposition 10. For the complete graph it is known ([17] eq. (7.3)) that D has Binomial$(n-2, 1/n)$ distribution: here Proposition 9 applies with $r = n-1$ and $a = \frac{1}{2}n/(n-1)$. For the cube graph Q_d one can calculate

$$a = \frac{1}{2}(1 - 1/d)/(1 - 2^{1-d}) \tag{9}$$

and deduce the distribution of D from the Proposition. Finally, recall that *a priori* $ED = 1 - 2/n$, so by taking expectations in Proposition 9 we get an expression for n in terms of (a, r). This suggests that a is perhaps determined by (n, r).

Proof of Proposition 9. Consider the random walk X_i generating the tree $\mathcal{T}$ as in Proposition 1. Let us suppose $X_0 \neq v$: we shall later indicate the modification for the case $X_0 = v$. Write $\Gamma(m)$ for the number of distinct neighbors of v visited through time m. Define $V's$ in terms of the random walk as follows.
(a) $V_0 = \Gamma(T_v)$.
$(b)V_1 = 0$ or 1 according as X_{T_v+1} was or was not previously visited.
(c) V_2 is the number of states j satisfying both

$$T_j > T_v + 1 \ ; \ X_{T_j - 1} = v.$$

Then the degree of v in $\mathcal{T}$ is $1 + V_1 + V_2$, so the issue is proving that the joint distribution of these $V's$ is as specified in Construction 8.

As in section 3 let ξ_i be the time of the i'th visit to the set A of neighbors of v. Fix i. Let S be the first time after ξ_i that the walk hits v or $A \backslash \{X_{\xi_i}\}$. There are three alternatives, with probabilities as stated.

$$\begin{array}{lll} (i) & \text{prob } = 1 - \psi & X_S = v. \\ (ii) & \text{prob } = \psi\frac{r - \Gamma(\xi_i)}{r-1} & X_S \in A, \Gamma(S) = \Gamma(\xi_i) + 1. \\ (iii) & \text{prob } = \psi\frac{\Gamma(\xi_i) - 1}{r-1} & X_S \in A, \Gamma(S) = \Gamma(\xi_i). \end{array}$$

Now suppose $\Gamma(\xi_i) = j$, say, and that v has not yet been hit. Then $\Gamma(\cdot)$ will reach $j+1$ before v is visited iff (ii) occurs before (i), and by the craps principle this has chance

$$\frac{\psi(r-j)/(r-1)}{1 - \psi + \psi(r-j)/(r-1)}.$$

This gives part (i) of Construction 8. Part (ii) is clear.

For part (iii), fix i, suppose $\xi_i > T_v$ and $\Gamma(\xi_i) = j$. Let S be the first time after ξ_i that the walk hits $A \backslash \{X_{\xi_i}\}$. Again there are three possibilities, with probabilities stated.

$$\begin{array}{rcl} (i) & \text{prob} = j/r & \Gamma(S) = j. \\ (ii) & \text{prob} = (1 - j/r)(1 - \theta) & \Gamma(S) = j + 1, X_{S-1} = v. \\ (iii) & \text{prob} = (1 - j/r)\theta & \Gamma(S) = j + 1, X_{S-1} \neq v. \end{array}$$

(The latter probabilities follow from (5)).

Now consider the first time S' that $\Gamma(\cdot)$ reaches $j + 1$. Then $X_{S'-1} = v$ iff (ii) occurs before (iii), and by the craps principle this has chance $1 - \theta$. This leads to part (iii) of Construction 8.

Finally, consider the case $X_0 = v$. I assert that the tree constructed by the walk $v = X_0, X_1, \ldots$ has the same degree as the tree constructed by $X_1, \ldots$, so this case is no different from the previous case. Let T^+ be the time of first return to v. Then the tree constructed by $X_0, \ldots, X_{T^+}$ is the same as the tree constructed by $X_1, \ldots, X_{T_v}$, except that the former has an edge (v, X_1) where the latter has an edge $(X_{T_v - 1}, v)$. The set of neighbors visited thus far is the same in each case, so the subsequent evolution of edges (v, w) is the same.

5 Asymptotics of degrees

For each K let $G = G^K, v = v^K$ satisfy Hypothesis 7. Let $D = D^K, r = r^K, a = a^K$, etc, be as in the previous section. Here we study the behavior of D^K as $K \to \infty$. For notational convenience we shall omit the superscripts K. Write $\mathcal{P}_\lambda$ for a Poisson(λ) r.v.:

$$P(\mathcal{P}_\lambda = i) = e^{-\lambda} \lambda^i / i!, \quad i \geq 0.$$

Suppose $r \to \infty$ and $a \to 1/2$. It is then straightforward to show that, for $\mathbf{V}$ as in Construction 8,

$$(V_0/n, V_1, V_2) \stackrel{d}{\to} (U_0, U_1, U_2)$$

where $\stackrel{d}{\to}$ denotes convergence in distribution, and where the limit distribution is described as follows.

(i) U_0 is uniform on [0,1] .
(ii) $U_1 = 0$ or 1; $P(U_1 = 1|U_0) = 1 - U_0$.
(iii) Given $U_0, U_1,$ U_2 has Poisson$(1 - U_0)$ distribution.
Now an easy transform argument shows

$$U_1 + U_2 \stackrel{d}{=} \mathcal{P}_1.$$

So Proposition 9 has the following asymptotic corollary.

Proposition 10 *Let $1 + D^K$ be the degree of v^K in the uniform spanning tree on G^K, where (v^K, G^K) is a sequence satisfying Hypothesis 7. Suppose $r^K \to \infty$ and $a^K \to 1/2$. Then $D^K \stackrel{d}{\to} \mathcal{P}_1$.*

For complete graphs (i.e. for the uniform random labelled tree) this is well known, and immediate from the exact Binomial distribution of D. On the cube graphs, (9) shows that Proposition 10 applies, and gives an apparently new result.

We suspect this asymptotic result holds without any explicit symmetry conditions.

Conjecture 11 *Let r^K be the degree of v^K in G^K. Let $1+D^K$ be the degree of v^K in the uniform spanning tree on G^K. For w in the neighbor-set A of v^K, let $\psi^K_w = P_w(T_{A \setminus w} < T_v)$. Suppose*

$$r^K \to \infty; \quad \sup_w |r^K(1 - \psi^K_w) - 1| \to 0.$$

Then $D^K \stackrel{d}{\to} \mathcal{P}_1$.

Now let us reconsider Corollary 6. Put $c(i, r) = i$. As $r \to \infty$, $ED^* \to \int_0^1 (1 - x)\, dx = 1/2$. Then the Poisson limit theorem for independent events shows $D^* \stackrel{d}{\to} \mathcal{P}_{1/2}$, giving the following result.

Corollary 12 *Let $1 + D^K$ be the degree of v^K in the uniform spanning tree on G^K, where G^K is a r^K-regular graph and $r^K \to \infty$. Then*

$$\liminf_K P(D^K \le u) \le P(\mathcal{P}_{1/2} \le u); u = 0, 1, 2, \ldots.$$

Conjecture 13 *Corollary 12 holds with $\mathcal{P}_{1/2}$ replaced by $\mathcal{P}_1$.*

This is closely related to open problems concerning covering by reversible Markov chains, which we now describe briefly. In fact, Corollary 6 shows that Conjecture 13 is true if Conjecture 14 below is true. For the latter could be applied to the random walk looked at only on the neighbor-set of v. Note also that, since $\mathrm{ave}_v D_v^K \to 1$, the truth of Conjecture 13 would imply that $D_v^K \to \mathcal{P}_1$ for "almost all" v.

Consider a Markov chain on r states with some *symmetric* transition matrix P. Let C_i be the number of distinct states visited during the first $i-1$ steps. On the complete graph (with $r = n-1$ large) we have $C_i \approx r(1-\exp(-i/r))$. To prove Conjecture 13 we want some result of the type "no reversible chain can hit more distinct states in time t than does the random walk on the complete graph". The following would suffice.

Conjecture 14 *Let $b(\varepsilon, r)$ be the maximum, over all r-state Markov chains with symmetric transition matrices, of*

$$P(C_i > (1+\varepsilon)r(1-\exp(-i/r)) \text{ for some } i).$$

Then $b(\varepsilon, r) \to 0$ as $r \to \infty$, for each $\varepsilon > 0$.

Known weaker results, in the context of the time taken to visit *all* states, are in [2].

6 Diameter of spanning trees.

Another natural quantity to study is the diameter Δ of the uniform random spanning tree $\mathcal{T}$. For fixed $r \geq 3$ it is easy to construct r-regular graphs G on n vertices such that *every* spanning tree has diameter $\Omega(n)$, or such that *every* spanning tree has diameter $O(\log n)$. Thus we cannot expect any interesting "universal" results, analogous to Proposition 5, for the diameter. For the uniform random labelled tree, we have [21]

$$\Delta/n^{1/2} \xrightarrow{d} L \text{ where } L \text{ has a certain nondegenerate distribution.} \qquad (10)$$

It seems likely that exactly the same result holds for some class of graphs including the cube graphs Q_d: this will be studied elsewhere. Here, we present only some crude bounds, in Theorem 15 below.

As before, let us consider only regular graphs G for simplicity. The transition matrix P for the random walk is symmetric, and so has real eigenvalues

$$1 = \lambda_1 > \lambda_2 \geq \lambda_3 \ldots.$$

Call $\tau = -1/(1-\lambda_2)$ the *relaxation time*. Work on random walks on graphs has produced many results of the form "a result for the complete graph remains approximately true on graphs G for which τ is suitably small relative to n": see e.g. [1, 8, 19]. Theorem 15 shows that, if $\tau = o(n^\varepsilon)$ for all $\varepsilon > 0$ then $\Delta = \Omega(n^{1/2-\varepsilon})$ and $O(n^{1/2+\varepsilon})$ for all $\varepsilon > 0$.

Let K denote an absolute constant, not necessarily the same from line to line.

Theorem 15 *Let Δ be the diameter of the uniform random spanning tree in a regular graph G on n vertices. Let τ be the relaxation time, as above. Then*

$$K^{-1}\tau^{-1}n^{1/2}(\log n)^{-1} \leq E\Delta \leq K\tau^{1/2}n^{1/2}\log n.$$

Let us make two vague conjectures.

(i) $\Delta = \Omega(n^{1/2})$ on graphs which are not too tree-like.

(ii) $\Delta = O(n^{1/2})$ on graphs where τ is polynomial in $\log n$ and where the chance of the random walk returning locally to its origin is bounded away from 1 (more precisely, where $a(G)/n$ is bounded above, for $a(G)$ as in Corollary 20).

The argument rests on the two lemmas below, concerning the random walk X_j on a regular graph, started uniformly. The first is a routine consequence of spectral theory; the second is a specialization of Theorems 5 ($\tau_4 \leq K\tau_1$) and 8 of [**?**].

Lemma 16 *For $1 \leq a \leq b$,*

$$\sum_{m=a}^{b} P(X_m = X_0) \leq \; b/n + K\tau n e^{-a/\tau}.$$

Lemma 17 *For all initial distributions ρ and all $B \subset G$,*

$$P_\rho(T_B > K\tau n\log(n)/|B|) \; \leq 1/2.$$

Proof of Theorem 15. As usual, we consider the spanning tree $\mathcal{T}$ constructed by the random walk X_j, as in Proposition 1. Suppose X_0 is uniform on G. Write $d(v,w)$ for the distance in $\mathcal{T}$ from v to w.

To establish the lower bound, fix $1 \leq a < b$. Let Z be the number of pairs (l,m) such that

$$0 \leq l < m \leq b, \quad m - l > a, \quad X_m = X_l.$$

If $Z = 0$ then, for each $i \leq b$, the path in $\mathcal{T}$ from X_i to X_0 starts with an edge (X_i, X_j) for some $i > j \geq i - a$. So $d(X_b, X_0) \geq b/a$. Thus

$$\begin{aligned}
P(\Delta < b/a) &\leq P(d(X_b, X_0) < b/a) \\
&\leq P(Z > 0) \\
&\leq EZ \\
&\leq b \sum_{m=a}^{b} P(X_m = X_0) \\
&\leq b^2/n + bK\tau n e^{-a/\tau} \text{ using Lemma 16.} \qquad (11)
\end{aligned}$$

To prove the lower bound we may suppose that n is arbitrarily large and that $\tau = o(n^{1/2}/\log n)$. Put $b =$ the integer part of $n^{1/2}/2$, and $a =$ the integer next larger than $\tau \log(2n^{3/2}K\tau)$. Then for sufficiently large n we have $1 \leq a < b$ and $a \leq K\tau \log n$. Then inequality (11) gives

$$P(\Delta < b/a) \leq 1/2$$

and so

$$\begin{aligned}
E\Delta &\geq (1/2)(b/a) \\
&\geq n^{1/2}/(K\tau \log n)
\end{aligned}$$

which is the desired lower bound.

To establish the upper bound, fix v_0. Let $v_0, v_1, \ldots, v_c$ be a path in G. Let B be the set $\{v_0, \ldots, v_c\}$. Let B_p be the event "the path in $\mathcal{T}$ from v_0 to the root X_0 starts out as $v_0, v_1, \ldots, v_c$". The argument rests on the following remarkable identity, whose proof we defer. Recall $T_B^+ = \min\{n \geq 1 : X_n \in B\}$.

Lemma 18

$$P(T_{v_c} = l | B_p) = \frac{P_{v_c}(T_B^+ > l)}{E_{v_c} T_B^+}, \; l = 0, 1, 2, \ldots.$$

Now define

$$s = \text{ integer next larger than } K\tau n \log n/(c+1).$$

Then Lemma 17 shows

$$P_\rho(T_B \geq s) \leq 1/2 \text{ for all distributions } \rho.$$

Then by iterating,

$$P_{v_c}(T_B^+ > js) \leq (1/2)^{j-1} P_{v_c}(T_B^+ > s)$$

for $j = 1, 2, \ldots$. Applying Lemma 18 with $l = js$ and again with $l = s$,

$$\begin{aligned} P(T_{v_c} = js | B_p) &\leq (1/2)^{j-1} P(T_{v_c} = s | B_p) \\ &\leq (1/2)^{j-1} s^{-1} \end{aligned}$$

using the fact that the right quantity in Lemma 18, and hence the left quantity, is decreasing in l. Using this decreasing property again,

$$P(js \leq T_{v_c} < (j+1)s | B_p) \leq (1/2)^{j-1}$$

and so, summing,

$$P(js \leq T_{v_c} | B_p) \leq (1/2)^{j-2}.$$

Given event B_p occurs, then $d(v_0, X_0) = c + d(v_c, X_0) \leq c + T_{v_c}$. Conversely, if $d(v_0, X_0) > c$ then B_p must occur for some unique path $v_0, \ldots, v_c$. Therefore we obtain

$$P(d(v_0, X_0) > c + js) \leq (1/2)^{j-2}.$$

But $\Delta/2 \leq \max_v d(v, X_0)$, and so

$$P(\Delta/2 > c + js) \leq n(1/2)^{j-2}.$$

It is now routine to show

$$\begin{aligned} E\Delta &\leq 2c + Ks \log n \\ &\leq 2c + K\tau n \log^2 n / c \end{aligned}$$

from the definition of s. Choosing $c =$ integer part of $\tau^{1/2} n^{1/2} \log n$ gives the desired upper bound.

Proof of Lemma 18. We may write

$$\{T_{v_c} = l\} \cap B_p = \{T_{v_c} = l = T_B\} \cap (\bigcap_{i=0}^{c-1} D_i),$$

where

$$D_i = \{T_{\{v_0, v_1, \ldots, v_i\}} = T_{v_i}, X(T_{v_i} - 1) = v_{i+1}\}.$$

The key fact is that, using the Markov property, the conditional probability $P(\bigcap D_i | T_{v_c} = T_B = l)$ does not depend on l. It follows that we can write

$$P(T_{v_c} = l | B_p) = \alpha P(T_{v_c} = T_B = l), \; l = 0, 1, 2, \ldots$$

where α does not depend on l. But

$$\begin{aligned}
P(T_{v_c} = T_B = l) &= \sum_w P(X_0 = w, T_{v_c} = T_B = l) \\
&= \sum_w P(X_0 = w, X_l = v_c, X_i \in G\backslash B \text{ for } 0 \le i \le l-1) \\
&= \sum_w P(X_0 = v_c, X_l = w, X_i \in G\backslash B \text{ for } 1 \le i \le l) \\
&= P_{v_c}(T_B^+ > l)
\end{aligned}$$

where the third identity uses time-reversal for the stationary random walk. This establishes the Lemma, up to the normalizing constant in the denominator, but this must be

$$\sum_{l=0}^{\infty} P_{v_c}(T_B^+ > l) = E_{v_c} T_B^+.$$

7 Counting Spanning Trees

The matrix-tree theorem gives a complicated expression for the number $N(G)$ of spanning trees in an arbitrary graph G. For r-regular G, there is an expression involving only the eigenvalues $(\lambda_2^{'}, \ldots, \lambda_n^{'})$ of the adjacency matrix ([10] Proposition 1.4 and Section 7.6), which can then be expressed in terms of the eigenvalues $(\lambda_2, \ldots, \lambda_n)$ of P, as follows.

$$N(G) = n^{-1} r^{n-1} \prod_{i=2}^{n} (1 - \lambda_i). \tag{12}$$

McKay [16] gives good upper bounds on $N(G)$, implicitly using random walk ideas, and in [15] discusses the case of random regular graphs.

Proposition 1 yields a probabilistic alternative to (12): pick some simple tree t and calculate the probability that the random walk construction gives t. The simplest tree is a path.

Proposition 19 *Suppose $v_1, v_2, \ldots, v_n$ is a Hamiltonian path in G. Let $B_i = \{v_1, \ldots, v_i\}$. Let ρ_i be the probability that the random walk on G started at v_i will return to v_i before exiting B_i. Then*

$$N(G) = \prod_{i=1}^{n-1} r_{v_i} (1 - \rho_i).$$

Proof. Let t be the path $v_1, \ldots, v_n$, considered as a tree rooted at v_1. Use the random walk started at v_1 to construct the random tree $\mathcal{T}$. Then

$$P(\mathcal{T} = t) = \prod_{i=1}^{n-1} P_{v_i}(\text{ exit } B_i \text{ along } (v_i, v_{i+1}))$$

where the "exit" event means that the random walk, when it first exits the subset B_i (which may or may not occur on the first step), does so along the edge (v_i, v_{i+1}). Then Proposition 1 implies

$$N(G) = \prod_{i=1}^{n-1} 1/P_{v_i}(\text{ exit } B_i \text{ along } (v_i, v_{i+1})). \tag{13}$$

Starting from v_i, the mean number of visits to v_i before exiting B_i is $1/(1-\rho_i)$, and each visit has chance $1/r_{v_i}$ to be followed by traversal of (v_i, v_{i+1}). So

$$P_{v_i}(\text{ exit } B_i \text{ along } (v_i, v_{i+1})) = \frac{1}{r_{v_i}} \frac{1}{1-\rho_i}$$

and the result follows.

It is not clear if Proposition 19 is ever useful for counting $N(G)$: presumably, if one has enough structure to calculate or bound the ρ_i then one has enough structure to calculate or bound the eigenvalues and use (12), at least in the regular case. On the other hand, having two conceptually different ways of calculating the same quantity is, throughout mathematics, often useful for some purpose or other. We end by pointing out two consequences of (12) and Proposition 19.

Markov chain theory shows that on a regular n-vertex graph G the following are equal - call the common value $a(G)$.
(i) $\sum_{k=2}^{n} \frac{1}{1-\lambda_k}$ where λ_k are the eigenvalues of P.
(ii) $n^{-1} \sum_w E_v T_w$ for each fixed v
(iii)$\sum_v \sum_{i=0}^{\infty} (P_v(X_i = v) - 1/n)$.
Moreover $a(G) \geq n-1$, with equality iff G is the complete graph. Because of (ii) we interpret $a(G)$ as the "average first hitting time" for the random walk on G. Putting together (i),(12) and the "geometric mean $\leq$ arithmetic mean" inequality gives

Corollary 20 *For a r-regular graph G on n vertices,*

$$(N(G))^{\frac{1}{n-1}} \geq \frac{rn^{\frac{1}{n-1}}(n-1)}{a(G)}.$$

Thus we can relate $N(G)$ to a natural property of the random walk on G.

Our second observation concerns a specific graph, the $K \times K$ discrete torus G_K. Consider the growth exponent

$$c = \frac{1}{4} \lim_K \, (N(G_K))^{1/K^2}.$$

Using (12) and the well-known eigenvalues for the simple random walk on the discrete torus we obtain (c.f. [10] Section 7.8 Exercise 10)

$$c = \exp(\int_0^1 \int_0^1 \log(\sin^2 \pi x + \sin^2 \pi y) dx dy) \approx 0.825.$$

On the other hand, by applying Proposition 19 with the natural Hamiltonian path

$$(1,1), (1,2), \ldots, (1,K), (2,K), (2,K-1), \ldots, (2,1), (3,1), (3,2), \ldots,$$

and approximating the random walk on G_K by simple symmetric random walk Y_j on the *infinite* square lattice $\mathbf{Z}^2$, we can obtain the following expression for c.

$$c = P_{(0,0)}(Y_j \text{ does not return to } (0,0) \text{ before exiting } B), \text{ where}$$

$$B = \{(i,j) : j < 0\} \cup \{(i,0) : i \leq 0\}.$$

Thus we have used spanning trees to evaluate a (non-trivial) probability associated with two-dimensional random walk!

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