15-859N — Spectral Graph Theory and Numerical Linear Algebra. — Fall 2011

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Assignment 4 Due date: Monday Nov 21, 2011

Please turn in each problem on a separate sheet of paper for grading purposes. Each sheet should include your name.

1 Construction of Low Stretch Spanning Trees

[5+5+5=15 points]

This problem tries to show an algorithm for finding a spanning tree with $O(m^{1/2}\log^2 n)$ average stretch in an unweighted graph with m edges.

1. Suppose we have a BFS tree of the graph starting at some vertex. Let E_l be the number of edges going from vertices at distance l to $l+1$ and B_l be the number of edges between vertices at distances $[0, l]$. Show that there exist $l <= m^{1/2}\log n$ such that $E_l \leq m^{-1/2}B_l$.
2. Show that in an unweighted graph, we can decompose it into pieces of radius $O(m^{1/2}\log n)$ so that at most $m^{1/2}$ edges are between the pieces in $O(m)$ time.
3. Apply the above procedure twice to give an algorithm with running time $O(m)$ that generates a spanning tree with average stretch $O(m^{1/2})$.

2 2-level Recursive Solve

[5+7+3=15 points]

We analyze a 3-level recursive preconditioner in this problem. You may assume any result about Chebyshev polynomials or conjugate gradient shown in class or otherwise.

The setup is as follows: we have 3 matrices A, B and C such that:

$$B \preceq A \preceq \kappa_1 B$$
$$C \preceq B \preceq \kappa_2 C$$

And we want to solve $Ax = b$ by only evaluating C^+

1. Observe that Preconditioned Chebyshev generates a linear operator. Show that one can find a linear operator $\textsc{PreconCheby}(B, C)$ such that:

$$1/2B^+ \preceq \textsc{PreconCheby}(\text{B, C}) \preceq 2B^+$$

   Such that $\textsc{PreconCheby}(B, C)$ can evaluated in $O(\sqrt{\kappa_2})$ forward multiplies in B and solves in C^+.

2. Show that PreconCheby$(B, C)^+$ can be used to precondition A, and give a bound on overall the iteration count in terms of κ_1, κ_2 and the error in A norm, ϵ. The bound should have terms corresponding to forward multiplies in A, B, and solves in C.

3. Argue that preconditioned conjugate gradient can be used in the preconditioned solves at the topmost level. Why can't this be done at the second level?

3 A Bad Example for Spectral Partitioning

[6+6+8=20 points]

Define **threshold spectral partitioning** of a possibly weighted graph $G = (V, E)$ to be the vertex partition one gets by; 1) Finding the eigenvector x for λ_2, 2) Sorting the vertices by their value in x, and 3) Returning the best threshold cut.

Let P_n^ϵ, for n even, be the weighted path graph on n vertices where all the edges have unit weight except the middle one which has weight ϵ.

Consider the Cartesian product $M_{cn}^\epsilon = P_{c\sqrt{n}} \otimes P_{\sqrt{n}}^\epsilon$

1. Show that threshold spectral partitioning on the graph M_{cn}^ϵ will generate a quotient cut of size $\Omega(1/\sqrt{n})$ for c sufficiently large and $\epsilon = 1/\sqrt{n}$ while the best cut is of size $O(1/n)$.

2. How big does c need to be for this to happen?

3. Show how to extend these results to the case when one uses the eigenvector from the normalized Laplacian.