15-859N — Spectral Graph Theory and Numerical Linear Algebra. — Fall 2011

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Assignment 3 Due date: Monday October 31, 2011

Please turn in each problem on a separate sheet of paper for grading purposes. Each sheet should include your name.

1 Chebyshev Recurrence

[10 points] In Class we defined the Chevbyshev polynomial to be:

$$T_0(x) = 1; \quad T_1(x) = x; \quad T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x) \text{ for } x \geq 1$$

The polynomial we actually need was:

$$\bar{T}_n(x) = T_n(x/\gamma/)/T_n(1/\gamma) \text{ for a fixed } 0 < \gamma < 1$$

Write a two term recurrence for $\bar{T}_n(x)$. Coefficients in your recurrence may include $T_i(1/\gamma)$

2 Chebyshev Polynomials

[25 points]

In this problem we will develop some important identities for Chebyshev Polynomials.

1. Consider the following matrix:

$$A_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

   Give a one sentence explanation why $A(\theta)$ is rigid counter clockwise rotation by θ degrees and A_θ^n is a rotation by $n\theta$ degrees.

2. We can abstract A_θ to a matrix $A = \begin{pmatrix} c & -s \\ s & c \end{pmatrix}$ where c and s are variables in some polynomial ring such that $c^2 + s^2 = 1$. Show that

$$A^n = \begin{pmatrix} T_n(c) & -sQ_n(c) \\ sQ_n(c) & T_n(c) \end{pmatrix}$$

   where T_n and Q_n are polynomials in c satisfying:

$$\begin{aligned} T_0(c) &= 1 \\ T_1(c) &= c \\ T_{n+1}(c) &= cT_n(c) - (1 - c^2)Q_n(c) \end{aligned}$$

and

$$\begin{aligned} Q_0(c) &= 0 \\ Q_1(c) &= 1 \\ Q_{n+1}(c) &= cQ_n(c) + T_n(c) \end{aligned}$$

3. Use these identities to show that:

$$\begin{aligned} T_{n+1}(c) &= 2cT_n(c) - T_{n-1}(c) \\ Q_{n+1}(c) &= 2cQ_n(c) - Q_{n-1}(c). \end{aligned}$$

   Thus T and Q are Chebyshev Polynomials of the first and second kind respectively. Explain why all the roots of T_n and Q_n lie in the interval $[-1, +1]$ and in this interval T and Q return values in this interval.

4. Show how to diagonalize A for $|c| \geq 1$.

5. Use this diagonal form to show that

$$T_n(c) = \frac{(c + \sqrt{c^2-1})^n + (c - \sqrt{c^2-1})^n}{2} = \frac{(c + \sqrt{c^2-1})^n + (c + \sqrt{c^2-1})^{-n}}{2}$$

3 Eigenvalues of Cartesian Products

[15 points]

Let $G = (V, E, w)$ and $H = (V', E', w')$ be two non-negatively weight graphs. Let $G \otimes H = (\bar{V}, \bar{E}, \bar{w})$ be their Cartesian product, where:

- The vertices are $\bar{V} = V \times V'$
- The edges are $\bar{E} = \{< (x, x'), (y, y') > \quad | \quad [x = y \wedge (x', y') \in E'] \vee [x' = y' \wedge (x, y) \in E]\}$
- What should the edge weights be?

1. Show that the eigenvalues of $L_{G \otimes H}$ are the direct sum of those of L_G and L_H. That is if the eigenvalues of L_G are $\{\lambda_1, \ldots, \lambda_n\}$ and those of L_H are $\{\mu_1, \ldots, \mu_m\}$ the those of $L_{G \otimes H}$ are $\{\lambda_i + \mu_j \quad | \quad 1 \leq i \leq n 1 \leq j \leq m\}$

2. Show that the eigenvectors of $L_{G \otimes H}$ are the direct product of those of L_G and L_H.