15-859N — Spectral Graph Theory and Numerical Linear Algebra. — Fall 2011

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Assignment 2 Due date: Monday October 17, 2011

Please turn in each problem on a separate sheet of paper for grading purposes. Each sheet should include your name.

1 Random Walks and Trolls

[10 points]

Consider the following variant of the random walk problem: you still proceed around the vertices in a random walk in an undirected graph with Laplacian L, but there are trolls on each of the vertices. Specifically vertex u contains x_u trolls. Given the vector $\mathbf{u}$ of number of trolls on all vertices, give an expression for the expected number of trolls that you encounter when walking from s to t. (you may use $\mathbf{e}_u$ to denote the vector that's 1 at u and 0 everywhere else.

2 Another Proof of the Perron-Frobenius Theorem

[10 + 5 + 5 = 20 points]

We give another, more direct proof of a slightly weaker version of the Perron-Frobenius theorem. Instead of showing that the stationary distribution converges, we show that the average of all distribution converged. That is, let $x^{(1)}$ be any initial probability vector and define:

$$x^{(t+1)} = Ax^{(t)}$$

and the running average

$$y^{(t)} = \frac{1}{t} \sum_{1 \leq t' \leq t} x^{(t')}$$

1. Let $z^{(t)} = Ay^{(t)} - y^{(t)}$. Show that $z^{(t)} = \frac{1}{t}(x^{(t+1)} - x^{(1)})$
2. Show that $y^{(t)} = Bz^{(t)}$ where B is a linear transformation.
3. Conclude that $y^{(t)}$ converges to some limit.

3 Linear System Solve on Trees

[10 points]

Suppose we're given a matrix A where the non-zero structure forms a tree. Give a $O(n)$ time algorithm for solving systems of the form $Ax = b$.

4 Bounding Separators Sizes using Path Embeddings

[10 points]

Show that an separator for the d dimensional mesh with n nodes has size $\Omega(n^{(d-1)/d})$ for d fixed using path embeddings.
Hint: Give a lower bound for separators for the complete graph and use this to bound the mesh.