15-859N
9/23/11

Springs and Graph Laplacian's

Input: Graph of Springs (mattress)

$G = (V, E, k)$

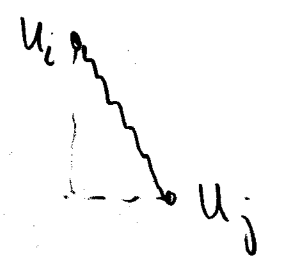
$k_{ij} \equiv$ spring constant for edge E_{ij}

Consider only vertical displacements

$u_i \equiv$ displacement of V_i

Goal: Find solutions to Newton: $f = ma$

Find forces for displacement $u = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$.



Set force on u_i by spring E_{ij}

$-(u_i - u_j) k_{ij}$

ie linear spring model.

$L = L(G)$

Force vector $-Lu$

Let $m_i \equiv$ mass of V_i $\quad M = \begin{pmatrix} m_1 & & 0 \\ & \ddots & \\ 0 & & m_n \end{pmatrix}$

$m_i > 0$

View u as a fcn of time $u(t) = \begin{pmatrix} u_1(t) \\ \vdots \\ u_n(t) \end{pmatrix}$

Acceleration: For $u_i(t) \equiv \dfrac{d^2 u_i}{dt^2}$

$$\frac{du}{dt} = \begin{pmatrix} \frac{du_1}{dt_1} \\ \vdots \\ \frac{du_n}{dt_n} \end{pmatrix} \quad \therefore \quad a = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \frac{d^2 u}{dt^2}$$

Newton: $-Lu = M\left(\dfrac{d^2 u}{dt^2}\right)$

two ways to solve:

1) Solve for u given initial conditions
say $u(0)$ & $u'(0)$

2) Find steady state solutions

Do 2) using Guess and check.

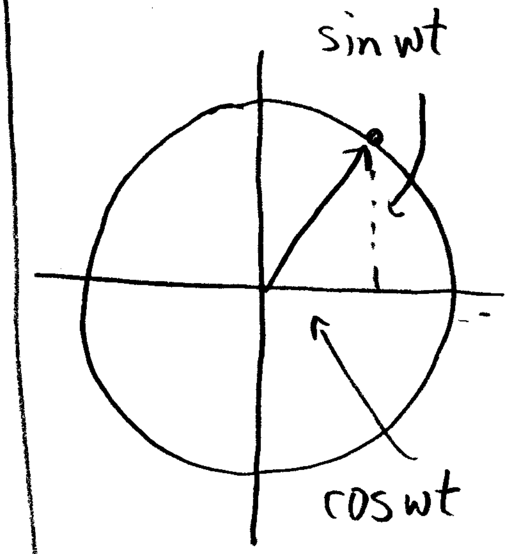
Recall: $e^{iwt} \equiv \cos wt + i \sin wt$

$$\frac{de^{iwt}}{dt} = iw\, e^{iwt} \overset{\text{or}}{\equiv} -w \sin wt + iw \cos wt$$

$$= iw(\cos wt + i \sin wt)$$

$$\frac{d^2 e^{iwt}}{dt^2} = \frac{d(iw\, e^{iwt})}{dt} = i^2 w^2 e^{iwt}$$

$$= -w^2 e^{iwt}$$



$$\frac{d \cos x}{dx} = -\sin x$$

$$\frac{d \sin x}{dx} = \cos x$$

Guess: $u = e^{i\omega t} x$ some vector x.

$$\frac{d^2 u}{dt} = -\omega^2 e^{i\omega t} x$$

Check:

$$M(-\omega^2 e^{i\omega t} x) \overset{?}{=} -L(e^{i\omega t} x)$$

iff $\omega^2 M x = L x$ set $\lambda = \omega^2$

iff $Lx = \lambda M x$ ie (λ, x) is a generlized eigen-pair.

Claim: Eigenvalues of $Lx = \lambda M x$ are real nonneg.

Change of variables $y = M^{1/2} x$ or $x = M^{-1/2} y$

$$L M^{-1/2} y = \lambda M M^{-1/2} y$$

$$\underbrace{M^{-1/2} L M^{-1/2}}_{L'} y = \lambda \cancel{M^{-1/2} M M^{-1/2}} y$$

L' is positive semidefinite

$\therefore$ we have found a space of dim n solutions.

eg $w_i = \sqrt{\lambda_i}$ then X_i the eigenvector

$$u(t) = \alpha_1 e^{iw_1 t} X_1 + \cdots + \alpha_n e^{iw_n t} X_n$$

is a solution.

<u>Consider</u> setting mass = the weighted degree

Eigenpairs $LX = \lambda D X$ $(*)$ $L = D - A$

Change of variable $Y = DX$

$(*)$ iff $(D-A)D^{-1}Y = \lambda Y$

$(I - AD^{-1})Y = \lambda Y$ $M = AD^{-1}$ transition matrix for a random walk.

iff $Y - MY = \lambda Y$

iff $-MY = (\lambda - 1)Y$ iff $MY = (1-\lambda)Y$

If $0 = \lambda_1 < \lambda_2 \cdots \leq \lambda_n$ eigenvalues of $(*)$

then $0 \leq 1 - \lambda_n, \ldots, 1 - \lambda_{n-1}, 0$ eigen of $MX = \lambda X$

Differential Eq

$$\frac{du}{dt} = Au \qquad A^T = A$$

$$\lambda(A) = \lambda_1 \le \lambda_2 \cdots \le \lambda_n$$

$$x_1 \quad \cdots \quad x_n$$

Guess & check $ce^{\lambda_i t} x_i = u$

$$\frac{du}{dt} = c\lambda_i e^{\lambda_i t} x_i$$

$$Au = A ce^{\lambda_i t} x_i = ce^{\lambda_i t} Ax_i$$

$$= ce^{\lambda_i t} \lambda_i x_i$$

$$u(t) = c_1 e^{\lambda_1 t} x_1 + \cdots c_n e^{\lambda_n t} x_n$$

$u(t)$ works!

Def $e^{A} \equiv I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots$

$$(e^{As})(e^{At}) = e^{A(s+t)} \qquad (e^{At})(e^{-At}) = I$$

$$\frac{d e^{At}}{dt} = A e^{At}$$

Suppose $A = S \Lambda S^{-1}$

$$e^{At} = I + S \Lambda S^{-1} t + \frac{S \Lambda^2 S^{-1} t^2}{2!} + \cdots$$

$$= S(I + \Lambda t \cdots) S^{-1} = S e^{\Lambda t} S^{-1}$$

Solve $\frac{du}{dt} = Au$ initial condition $u(0)$

Claim $e^{At}u(0)$ works

$$\frac{d}{dt} e^{At}u(0) = Ae^{At}u(0)$$

<u>Thm</u> $A^T = A \wedge B^T = B$ then $tr(e^{A+B}) \leq tr(e^A \cdot e^B)$

Trotter-Suzuki.

GOLDEN-THOMPSON INEQUALITY

For $n \times n$ complex matrices, the matrix exponential is defined by Taylor series as

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}.$$

For commuting matrices A and B we see that $e^{A+B} = e^A e^B$ by multiplying the Taylor series. This identity is not true for general non-commuting matrices. In fact, it always fails if A and B do not commute, see [2].

Theorem 1 (Golden-Thompson Inequality). *For arbitrary self-ajoint matrices A and B, one has*

$$\operatorname{tr}(e^{A+B}) \le \operatorname{tr}(e^A e^B).$$

For a survey of Golden-Thompson and other trace inequalities, see [2]. In the present note, we give a proof of Golden-Thompson inequality following [1] Theorem 9.3.7.

Remarks. 1. Golden-Thompson inequality holds for arbitrary unitary-invariant norm replacing the trace, see [1] Theorem 9.3.7.

2. A version of Golden-Thompson inequality for three matrices fails: $\operatorname{tr}(e^{A+B+C}) \not\le \operatorname{tr}(e^A e^B e^C)$.

The proof of Golden-Thompson inequality is based on Lie Product Formula:

Theorem 2 (Lie Product Formula). *For arbitrary matrices A and B, we have*

$$e^{A+B} = \lim_{N\to\infty} (e^{A/N} e^{B/N})^N.$$

Proof. We first compare

$$X_N = e^{(A+B)/N} \quad \text{and} \quad Y_N = e^{A/N} e^{B/N}.$$

As $N \to \infty$, Taylor's expansion gives

$$\begin{aligned} X_N &= 1 + \frac{A+B}{N} + O(N^{-2}), \\ Y_N &= \Big[1 + \frac{A}{N} + O(N^{-2})\Big]\Big[1 + \frac{B}{N} + O(N^{-2})\Big] \\ &= 1 + \frac{A}{N} + \frac{B}{N} + O(N^{-2}). \end{aligned}$$

This shows that

(1) $$X_N - Y_N = O(N^{-2}).$$

For a proof, see [1] Theorem 2.3.6.

Proposition 3 follows from Weyl's Majorant Theorem for the function $f(x) = x^m$:

$$|\operatorname{tr}(X^m)| = \left|\sum_{i=1}^{n} \lambda_i^m\right| \leq \sum_{i=1}^{n} |\lambda_i|^m \leq \sum_{i=1}^{n} s_i^m = \operatorname{tr}(|X|^m).$$

Proof of Golden-Thompson Inequality. Fix a natural number N and consider

$$X = e^{A/2^N}, \quad X = e^{B/2^N}.$$

To prove Golden-Thompson Inequality, it suffices to show that

$$\operatorname{tr}((XY)^{2^N}) \leq \operatorname{tr}(X^{2^N} Y^{2^N}). \tag{2}$$

Indeed, if (2) holds then, taking limit as $N \to \infty$ we see that the left hand side of (2) converges to $\operatorname{tr}(e^{A+B})$ by Lie Product Formula, while the right hand side equals $\operatorname{tr}(e^A e^B)$.

To prove (2), we use Proposition 3 and note that $|XY|^2 = (XY)^*(XY) = YX^2Y$. We thus have

$$\operatorname{tr}(XY)^{2^N} \leq \operatorname{tr}(YX^2Y)^{2^{N-1}} = \operatorname{tr}(X^2Y^2)^{2^{N-1}},$$

where the last equality follows from the trace property $\operatorname{tr}(UV) = \operatorname{tr}(VU)$.

Continuing this procedure for X^2 and Y^2, we obtain

$$\operatorname{tr}(X^2Y^2)^{2^{N-1}} \leq \operatorname{tr}(X^4Y^4)^{2^{N-2}}.$$

After N steps, we arrive at the bound (2). This proves Golden-Thompson Inequality. □

References

[1] R. Bhatia, *Matrix analysis.* Graduate Texts in Mathematics, 169. Springer-Verlag, New York, 1997. xii+347 pp.

[2] , D. Petz, *A survey of trace inequalities*, in Functional Analysis and Operator Theory, 287-298, Banach Center Publications, 30 (Warszawa 1994). Online at `http://www.renyi.hu/~petz/pdf/64.pdf`