15-859N
10/31/2011

Greedy Elim-1-2

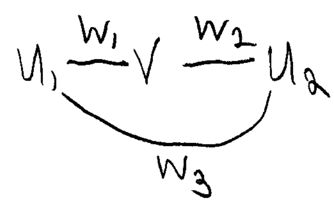
Goal 1) Use GE to remove all degree 1 & 2 nodes.
2) But keep spanning tree.

Greedy Elim

Input: $G=(V,E,w)$ & ST T of G

Output: $\hat{G}=(\hat{V},\hat{E},\hat{w})$ & ST $\hat{T}$ of $\hat{G}$

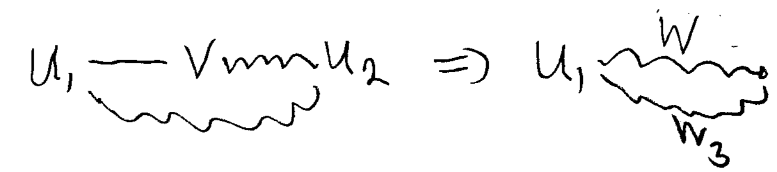
1) $\hat{G}:=G$; $E_{\hat{T}}=E_T$
2) while $\exists v$ of degree ≤ 2 do
   if degree$(v)=1$ then remove v.
   if degree$(v)=2$



— tree edges
~ non-tree edges

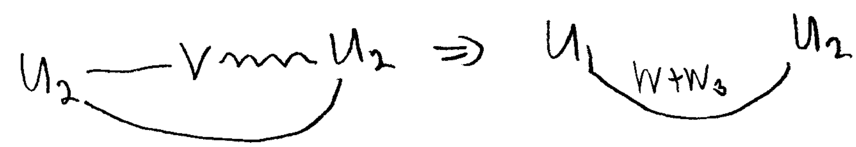
$$W \Leftarrow \frac{w_1 \cdot w_2}{w_1+w_2}$$

Case 1



(2 samples)

Case 2



Case 3 $\quad u_1 — v — u_2 \Rightarrow u_1 \xrightarrow{w+w_3} u_2$

Claim $\hat{T}$ is ST of $\hat{G}$ & $|\text{nontree}(\hat{G})| \le |\text{nontree}(G)|$

if $K = |\text{nontree}(G)| = K$ then $|\hat{V}| \le 2K$ & $|\hat{E}| \le 3K$

pf

Let n_1 = # leaves in $\hat{T}$

n_2 = # degree 2 node in $\hat{T}$

n_3 = # " " ≥ 3 " " .

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note $n_3 \le n_1$ thus $n = n_1 + n_2 + n_3 \le 2n_1 + n_2 \le 2K$

$\hat{G}$ has $2K-1$ tree edges & K nontree edges

$m \le 3K$

Claim $\text{Stretch}_T(e) \ge \text{Stretch}_{\hat{T}}(e)$

Gaussian Elim Details

Suppose $H \xrightarrow{GE} G$

ie $L_k \cdots L_1 H L_1^T \cdots L_k^T = \begin{pmatrix} I & 0 \\ 0 & G \end{pmatrix}$

$$H = L_1^{-1} \cdots L_k^{-1} \begin{pmatrix} I & 0 \\ 0 & G \end{pmatrix} L_k^{-1T} \cdots L_1^{-1T}$$

Goal Solve $Hx = b$

[project & solve] $\begin{pmatrix} I & 0 \\ 0 & G \end{pmatrix} L_k^{-1T} \cdots L_1^{-1T} x = L_k \cdots L_1 b = b'$

solve $\begin{pmatrix} I & 0 \\ 0 & G \end{pmatrix} y = b' \Rightarrow G y_G = b' \lceil G$

$$y = L_k^{-1T} \cdots L_1^{T} x$$

[expand] $x = L_1^T \cdots L_k^T y$

Building Preconditionin Chain

<u>Def</u> $C = \{G = G_1, H_1, G_2, \cdots, G_d\}$ chains of graphs.

$\mathcal{K} = \{\kappa_1, \ldots, \kappa_{d-1}\}$ list of numbers

$\{C, \mathcal{K}\}$ is a <u>good Precon Chain</u> if $\exists \{\mu_1, \cdots, \mu_d\}$ st.

1) $G_i \preceq H_i \preceq \kappa_i G_i$

2) $G_{i+1} = \text{GreedyElim}(H_i)$

3) $|G_i| = \mu_i$

4) $\mu_1, \mu_2 \leq m$

5) $\mu_i / \mu_{i+1} > \lceil c_r \sqrt{\kappa_i} \rceil \quad \forall i$

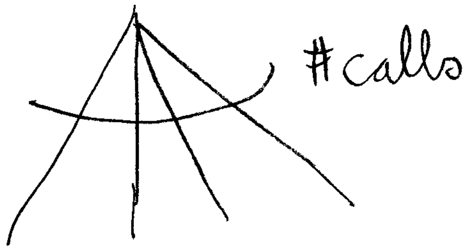
6) $\kappa_i \geq \kappa_{i+1}$

7) μ_d bded.

Determining Work

2-Level

Probs $G_i x = b$



$G_{i+1} x = b_1$ - - - - - - $G_{i+1} x = b_k$

#subcalls.

$\text{Level}_i \text{ work} \equiv O(|G_i| \cdot \#\text{calls})$

#calls
Matrix-Vector products

$\text{Level}_{i+1} \text{ work} \equiv$

$$O(\#\text{calls} \cdot |G_{i+1}| \cdot \#\text{subcalls})$$

Goal: $\dfrac{\text{Level}_{i+1}\text{work}}{\text{Level}_i\text{-work}} \leq c < 1$

$$\frac{\#\text{calls} \cdot |G_{i+1}| \cdot \#\text{subcalls}}{|G_i| \cdot \#\text{calls}} \leq \#\text{calls} \cdot \frac{|G_{i+1}|}{|G_i|} \qquad (6)$$

$$< \frac{\#\text{calls}}{c_r\sqrt{K_i}} = \frac{\sqrt{K_i}}{c_r\sqrt{K_i}} < \frac{1}{c_r} < 1$$