

15-859N
11/14/11

Symmetric Diagonally Dominant Systems

Let A SDD in $a_{ii} \geq \sum_{i \neq j} |a_{ij}|$

Goal: Reduce SDD to Laplacians!

First idea: Do a change of variables.

SDD $\equiv \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ to solve $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$

Change of variables $x' = -x$ & $a' = -a$

Solve $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -x' \\ y \end{pmatrix} = \begin{pmatrix} -a' \\ b \end{pmatrix}$

in $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} x' \\ y \end{pmatrix} = \begin{pmatrix} a' \\ b \end{pmatrix}$

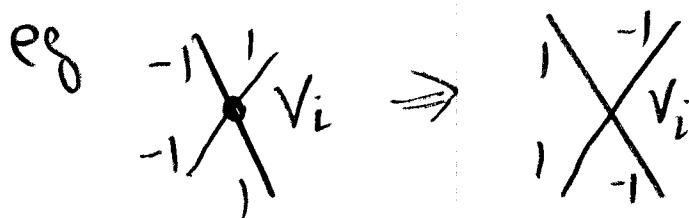
But this is:
$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x' \\ y \end{pmatrix} = \begin{pmatrix} a' \\ b \end{pmatrix}$$

SDD $\equiv$ Graph Laplacians with neg edge weights.

$$G = (V, E, w) \quad w(e) \in \mathbb{R}$$

Change of variables $x'_i = -x_i$ & $b'_i = -b_i$

Same as negating all edges at V_i



Goal: Make all weights positive by a change of variables.

Let C be a cycle or path in G

Def
$$\text{sign}(C) = \prod_{e \in C} \text{sign}(w(e))$$

Thm All cycles of G have pos sign iff

$\exists$ change of variables with positive edge weights.

$(\Leftarrow)$ The sign of a cycle is unchange by COV.

$(\Rightarrow)$ Assume G is connected

pick a vertex $x \in V$

$$p(y) = \text{sign}(P_{xy})$$

Claim $p(y)$ is well-defined

Let P_{xy} & P'_{xy} be 2 paths from x to y .

$P_{xy} * P'_{yx}$ is a cycle

$$1 = \text{sign}(P_{xy} * P'_{yx}) = \text{sign}(P_{xy}) \cdot \text{sign}(P'_{yx})$$

$$\Rightarrow \text{sign}(P_{xy}) = \text{sign}(P'_{xy})$$

If All cycles of G are positive.

we do a COV and solve system.

Assume G connected and some neg cycle.

We construct a 2-fold cover

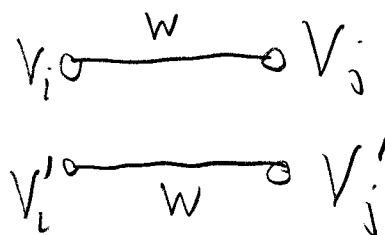
Double Cover

Input: $G = (V, E, W)$

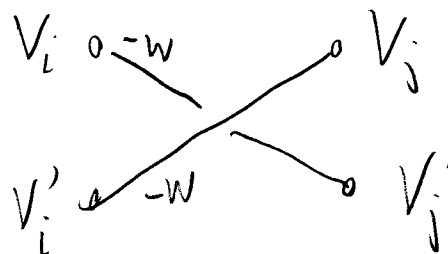
1) $\bar{V} = \{V_1, V_1', V_2, \dots, V_n\}$

2) for each edge $\overset{V_i}{\circ} \xrightarrow{w} \overset{V_j}{\circ}$

a) If $w > 0$ add edges



b) If $w < 0$ add edges



Return $\bar{G} = (\bar{V}, \bar{E})$

Claim G connected with neg cycle then
 $\bar{G}$ is connected.

pt Suffice to show that v_i connected to v_i' .

iff neg cycle containing v_i

Let C be a neg cycle & P a path from v_i to C

Claim: P, C, P^R is neg.

Cor $L(\bar{G})$ has rank $2n-1$.

$$G^+ = (V, E^+, w^+)$$

$$e \in E^+ \text{ iff } e \in E \text{ \& } w(e) > 0$$

$$G^- = (V, E^-, w^-)$$

$$A^+ \text{ adj matrix } G^+$$

$$A^- \text{ " " } G^-$$

$$\bar{A} = \begin{pmatrix} A^+ & A^- \\ A^- & A^+ \end{pmatrix} \text{ adj matrix } \bar{G}$$

$$L(\bar{G}) = \begin{pmatrix} D & 0 \\ 0 & D \end{pmatrix} - \bar{A} = \bar{L}$$

Claim $Lx = b \iff L \begin{pmatrix} x \\ -x \end{pmatrix} = \begin{pmatrix} b \\ -b \end{pmatrix}$

Claim $L \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b \\ -b \end{pmatrix} \text{ \& } \begin{pmatrix} x \\ y \end{pmatrix}^T \underline{1} = 0 \text{ then } y = -x$