

Resistive Theory

15-859N
9/14/11

Network (Resistive)

undirected graph $G=(V,E)$ (oriented)

conductance $C:E \rightarrow \mathbb{R}^+$

Flow (Kirchhoff flow) from a to b .

$j:E \rightarrow \mathbb{R}$ st

1) $j_{xy} = -j_{yx}$

2) $\sum_y j_{xy} = 0$ if $x \neq a, b$

3) $j_{xy} = 0$ if $(x,y) \notin E$

Def j is an Ohms' flow if

$\exists V:V \rightarrow \mathbb{R}$ st

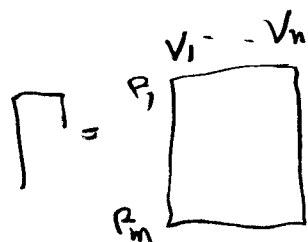
$$j_{xy} = C_{xy}(V_x - V_y) \quad (x,y) \in E$$

Def Residual flow $i_x = \sum_y i_{xy}$

Graph Laplacians

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$$L_G = L = D - A \quad D = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix} \quad A = \text{Adj}$$



$$L = \Gamma^T C \Gamma \quad C = \begin{pmatrix} c_1 & & 0 \\ & \ddots & \\ 0 & & c_m \end{pmatrix}$$

Thm If $V = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$ Voltages with residual currents $i = \begin{pmatrix} i_1 \\ \vdots \\ i_n \end{pmatrix}$
 then $LV = i$

Energy / Power

If $V \equiv$ voltages then $V^T L V = \sum c_{xy} (V_x - V_y)^2 \equiv \text{Energy}$

Assume that G is connected $\wedge L_G = L$.

$$x^T L x = 0 \text{ iff } \Gamma x = 0 \text{ iff } x \perp 1 = 0$$

connected

$$Lx = 0 \Rightarrow x^T L x = 0 \Rightarrow x \perp 1 = 0 \Rightarrow Lx = 0$$

$$\text{Rank}(\text{Ker}(L)) = 1 \Rightarrow \text{Rank}(L) = n - 1$$

Suppose $a, b \in V$ & effective resistance R_{ab}

Def Effective energy $i_{ab}^2 R_{ab} = R_{ab}$ if $i_{ab} = 1$

Real energy is:

$$\text{solve } LV = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ -1 \end{pmatrix} \quad V_a = V_1 \text{ \& } V_b = V_n$$

$$\text{Energy } V^T L V = V^T \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ -1 \end{pmatrix} = V_a - V_b$$

Thm Real energy = effective energy

Conservation of Energy

Let $W \equiv$ any voltage settings
 $j \equiv$ any flow from a to b.

Then $(W_a - W_b) j_a = \frac{1}{2} \sum_{x,y} (W_x - W_y) j_{xy}$

$$2RHS = \sum_{x,y} (W_x - W_y) j_{xy} = \sum_x W_x \sum_y j_{xy} - \sum_y W_y \sum_x j_{xy}$$

$$= W_a \sum_y j_{ay} + W_b \sum_y j_{by} - \left(W_a \sum_x j_{xa} + W_b \sum_x j_{xb} \right)$$

$$= W_a j_a + W_b j_b - W_a (-j_a) - W_b (-j_b)$$

$$= W_a j_a - W_b j_a + W_a j_a - W_b j_a$$

$$= 2(W_a - W_b) j_a$$

Thomson's Principle

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i & j $\equiv$ unit flows from a to b

i satisfies Ohm's Law then

$$\sum i_{xy}^2 R_{xy} \leq \sum j_{xy}^2 R_{xy}$$

pf let $d = j - i$ then d is a zero flow from a to b .

$$\sum j_{xy}^2 R_{xy} = \sum (i_{xy} + d_{xy})^2 R_{xy}$$

$$= \sum i_{xy}^2 R_{xy} + 2 \sum i_{xy} R_{xy} d_{xy} + \sum d_{xy}^2 R_{xy}$$

$$+ 2 \sum (V_x - V_y) d_{xy} \quad (*)$$

Claim $(*) = 4(V_a - V_b) d_a = 0$ by

conservation of energy.

thws
$$\sum j_{xy}^2 R_{xy} = \sum i_{xy}^2 R_{xy} + \sum d_{xy}^2 R_{xy}$$
$$\geq \sum i_{xy}^2 R_{xy}$$

Rayleigh's Monotonicity Law

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Network $G = (V, E, c)$ $\bar{G} = (V, \bar{E}, \bar{c})$

If $\forall x, y \in V$ $\bar{R}_{xy} \geq R_{xy}$ then $\bar{E}R_{ab} \geq ER_{ab}$

pf need two Thms first.

Let $i \equiv$ unit Ohms flow from a to b in $\bar{G}$
"6
L^A

$$\bar{E}R_{ab} = 1^2 \bar{E}R_{ab} \quad \text{effective energy}$$

$$= \frac{1}{2} \sum i_{xy}^2 \bar{R}_{xy} \quad \text{real energy}$$

$$\geq \frac{1}{2} \sum i_{xy}^2 R_{xy} \quad \text{Hypothesis}$$

$$\geq \frac{1}{2} \sum i_{xy}^2 R_{xy} \quad \text{Thompson}$$

$$= ER_{ab} \quad \text{effective}$$