15-859N
11/18/11

# Counting & Finding Spanning Trees

$G \equiv$ directed graph

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$H \subseteq G$ is <u>divergent ST</u> with root $r$ if

1) $r$ can reach all of $V(G)$ in $H$.
2) $\text{indegree}(v) = 1$ for $v \neq r$.
3) $\text{indegree}(r) = 0$

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<u>Def</u> $D$ is in-degree matrix of $G$ if

$$D(i,j) = \begin{cases} d_{in}(i) & \text{if } i = j \\ -k = -\#\text{edge from } i \text{ to } j & i \neq j \end{cases}$$

<u>Lemma</u> $G=(V,E)$ is a divergent Tree iff

1) $D(i,i)=\begin{cases} 0 & \text{if } i=r \\ 1 & \text{o.w} \end{cases}$

2) det of minor of $D_{r,r}$ (removing rth col & row) $=1$

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($\Rightarrow$) 1) Clear

2) After permuting row & col

$D$ is upper $\triangle$ & $r=1$

$\det(D_{1,1})=1$

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($\Leftarrow$) suppose false.

$\therefore$ G must contain a disconnected component C.

Each node of C has indegree 1 and zero col sums.

Thus $\det(D(C))=0$.

$\Rightarrow \det(D(G))=0$ contra!

**Thm** # divergent ST rooted at $r \equiv \det(D_{r,r})$

pf WLOG assume $r = v_1$

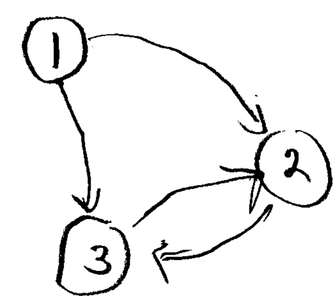
Note: Col sums of $D$ are zero.

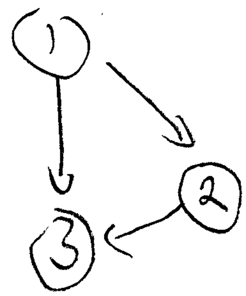
Start by replacing first col with $\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

$\det(D) = \det(D_{1,1})$.

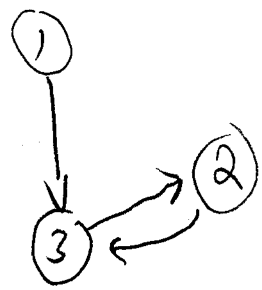
Expand col 2 into subgraphs with fix edge into $V_2$.

eg





$G_1$



$G_2$

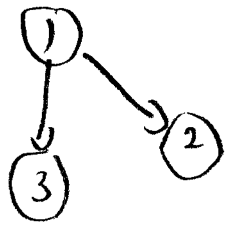
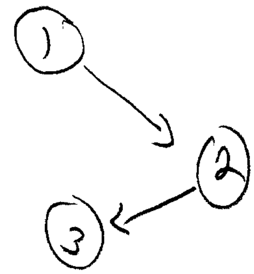
$$D = \begin{pmatrix} 1 & -1 & -1 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

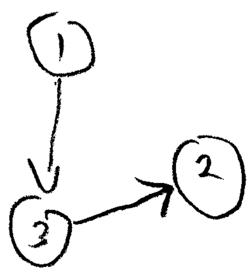
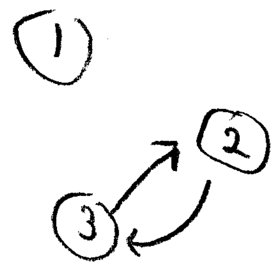
$$|D| = \begin{vmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & -1 & 2 \end{vmatrix}$$

For matrix expand next col.

Repeat!

We get exact one matrix per indeg 1 subgraphs.

eg $G_1 \overset{\text{expand}}{\Longrightarrow}$  + 

$G_2 \overset{\text{expand}}{\Longrightarrow}$  

Each term is $\begin{cases} 1 & \text{if } \equiv \text{ divergent tree} \\ 0 & \text{o.w.} \end{cases}$

$\square$

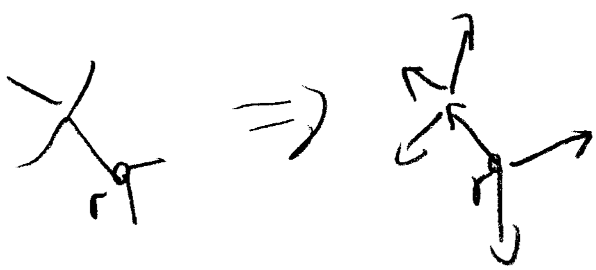
## The undirected Case

$G = (V, E)$ undirected

Let $G' = (V, E')$ where each edge $\{i, j\}$ is viewed as $(i,j)$ & $(j,i)$

<u>Claim</u> $\# ST(G) = \# ST_r(G')$

$f : ST(G) \to ST_r(G')$



1-1 & onto

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Note $D(G') \equiv$ Laplacian of $G$

$\mathcal{T}_i(G) \equiv ST(G)$ rooted at $i$

$\mathcal{T}(G) \equiv$ All rooted ST.

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<u>Markov Chain Thm</u> Let $\pi_i \equiv$ stationary prob of being at $i$

$$\pi_i = \frac{\sum_{T \in \mathcal{T}_i(G)} w(T)}{\sum_{T \in \mathcal{T}(G)} w(T)}$$

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pf $W = X_0, X_1, \ldots,$ a (random) walk.

<u>Def</u> $B_t \equiv$ (backwards tree at time $t$)

$I = \{X_0, \ldots, X_t\}$

$$\ell(i,T) = \operatorname*{argmax}_{0 \le j \le t} \{X_j = X_i\}$$

$$E(B_t) = \{(X_{\ell(i,t)}, X_{\ell(i,t)+1})\} \quad i \in I - X_t$$

root is $X_t$

# Random ST by Random Walks

**Alg** **WalkTree** Input: $G=(V,E)$ connected undirected (unit weight)

1) Pick $s \in V$ arbitrary
2) Do random walk from $s$
3) Collect edge $(i,j) \in E$ if first visit to $j$
4) Return tree

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**Thm** The WalkTree is random ST

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**Preliminaries**

Random walk on strongly connected directed graphs

ST $\equiv$ convergent rooted trees

$G=(V,P)$ $\quad \sum_j P_{ij}=1 \quad P_{ij}=W_{ij}$

$$w(T)=\prod_{e\in T} w(e)$$

| $t$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| $X$ | $X_2$ | $X_7$ | $X_1$ | $X_8$ | 2 | 8 | 1 | 8 | 2 | 8 |

$B_5$

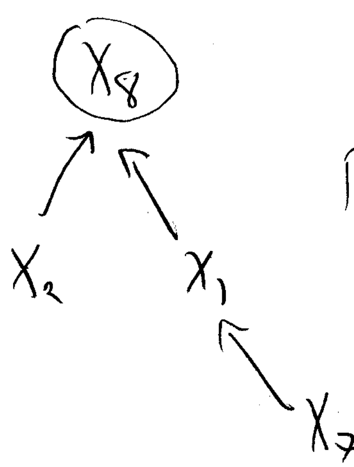
$l(0,5) = \emptyset$

$l(1,5) = 2 \qquad X_1 \to X_8$

$l(2,5) = 4 \qquad X_2 \to X_8$

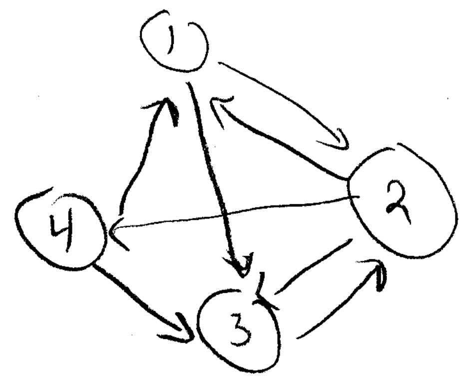
$l(7,5) = 1 \qquad X_7 \to X_1$

$l(8,5) = 5 \qquad \cancel{X_8 \to X_1}$




$B_5$

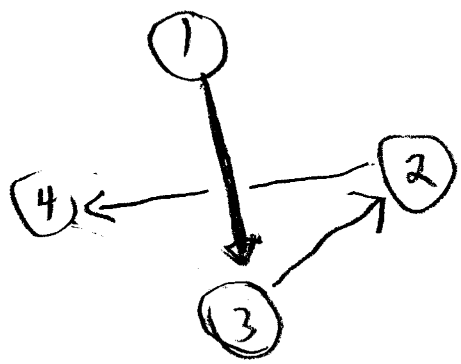
eg




Time 0 1 2 3 4 5

$W = (1, 2, 1, 3, 2, 4, 3 \ldots)$

$B_5$




root 4

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Note: $\{W\} \supseteq V$ then conver ST.

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After cover time we have a random walk on rooted trees of $G$

$B_t, B_{t+1}, B_{t+2}, \ldots$

One recurrent class

$\tau(T) \equiv$ stationary prob of $T$

<u>Note</u>

$$\pi_i = \lim_{N\to\infty} \frac{1}{N} \sum_{0\le t\le N} P_r(X_t = i)$$

$$= \lim_{N\to\infty} \frac{1}{N} \sum_t P_r(B_t \text{ rooted at } i)$$

$$= \sum_{T\in T_i} \tau(T)$$

<u>To show</u> $\tau(T) \approx w(T)$

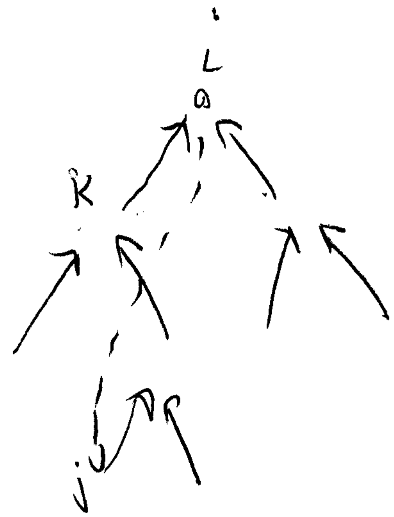
Let $T^{(i)} \equiv$ ST rooted at $i$

<u>Find</u> precursors of $T^{(i)}$!

$$T^{(k)} \xRightarrow[\text{step}]{\text{one}} T^{(i)}$$

$K$ child of $i$ and Parent$(i)$ in $T^{(k)}$ is in subtree of $K$ in $T^{(i)}$

$T^{(i)}$




$$T^{(k)} = T^{(i)} + [i,j] - [k_j, t]$$

$$\nabla(T^{(i)}) = \sum_{(i,j)\in E} \nabla\left(T^{(i)} + [i,j] - [k_j, i]\right) P_{k_j i}$$

$$\sum W\left(T^{(i)} + [i,j] - [k_j, i]\right) P_{k i}$$

$$= \sum \frac{W(T^{(i)})\, P_{ij}}{\cancel{P_{ki}}} \cancel{P_{ki}} = W(T^{(i)})$$

<u>Def</u> (Forward Tree) $F_t$

$$f(i,t) = \underset{0 \le j \le t}{\arg\min} \{ X_j = V_i \}$$

Edges $\left\{ \left( X_{f(i,t)}, X_{f(i,t)-1} \right) \middle| \ i \in I - X_0 \right\}$

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<u>Thm</u> Let $c$ = cover time for $G$, stationary $\equiv \pi_1 \cdots \pi_n$

$F_c \equiv$ forward tree at time $c$, $G$ is undirected

$$P_r(B_c = T) = P_r(F_c = T) = \frac{w(T)}{\sum_{T' \in T(G)} w(T')} \quad \left( \begin{array}{c} \text{starting from} \\ \text{stationary} \end{array} \right)$$

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<u>Pf</u> Since started from stationary

$$P_r(X_0 = V_0, \ldots, X_K = V_K) = P_r(X_0 = V_K, \ldots, X_K = V_0)$$

Thus $Pr(B_K = T \mid \pi) = Pr(F_K = T \mid \pi)$

pass to limit!

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Cor 1 M random walk on undirected $G = (V, E)$ starting at $i$

C is the cover time starting from $\pi$

Tree $F_C$ is uniform random ST rooted at $i$

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Pf $w(T) = d_i / \prod_{j \in V} d_j$ T rooted at $i$

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Cor 2 Hypo same as Cor 1 but we return

undirected $F_c$ then get random tree.

Pf 1-1 correspondence with ST.