15-859N
9/30/11

Perron-Frobenius Thm

Let $A^{n \times n} \geq 0$ & G is the incidence graph of A

Let $G^k \equiv$ the graph of pairs connected by a path of length k.

<u>Claim 1</u> The graph of $A^k = G^k$.

<u>Claim 2</u> The graph of $(I+A)^k = G^1 \cup \cdots \cup G^k$

<u>Def</u> A is <u>irreducible</u> if G is strongly connected.

<u>Claim 3</u> A irred $\Leftrightarrow (I+A)^{n-1} > 0$

<u>Def</u> $\rho(A) = \max\{|\lambda_i| : \lambda_i \in \lambda(A)\}$

<u>Thm</u> (Perron-Frobenius) $A \geq 0$ & irred then

a) $\rho(A)$ is a simple eigenvalue of A

If x is its eigenvector then $x > 0$ or $x < 0$

b) If $0 \leq A_1 \leq A$ then $\rho(A_1) \leq \rho(A)$

$\rho(A_1) = \rho(A)$ iff $A_1 = A$

c) $\theta \in \lambda(A)$ & $|\theta| = \rho(A)$ then

i) θ/ρ is mth root of unity

ii) $\forall r \quad (\theta/\rho)^r \theta \in \lambda(A)$

iii) All cycles have length mult of m

<u>Pf</u>

It will take the rest of the lecture.

Def x is λ-subharmonic for A if

$$0 \neq x \geq 0 \quad \& \quad Ax \geq \lambda x$$

An example Suppose $Ax = \theta x$ then $|x|$ is $|\theta|$-subharmonic for $|A|$ $\left(|A|_{ij} = |A_{ij}| \right)$

Pf $|\theta||x_i| = |\theta x_i| = |(Ax)_i| = |\sum_j A_{ij} x_j| \leq \sum_j |A_{ij}||x_j|$

Lemma 1 $A^{n\times n} \geq 0$ irred then

a) $\{\rho \mid \exists x \in R^n \text{ s.t } x \text{ is } \rho\text{-subharmonic}\}$ is closed & compact.

b) If ρ is a max then
   i) $Ax \geq \rho x \Rightarrow Ax = \rho x$
   ii) $x > 0$

proof of lemma

Def $F(x) = \min_{\substack{i \\ x_i \neq 0}} \frac{(Ax)_i}{x_i} = \max_{\rho} (Ax \geq \rho x)$

$S = \{x \in \mathbb{R}^n \mid x \geq 0 \ \& \ 1^T x = 1\}$

note S is closed & compact set.

Goal: Set $\rho = \max_{x \in S} F(x)$.

Prob F is not continous at ∂S!

Def $T = (I + A)^{n-1} S$

Claim a) T is closed & compact.

b) F is continous on T.

c) $\exists z \in T$ s.t. $F(z) = \rho$ is maximized pf of claim

Let $B = (I+A)^{n-1}$.

a) Since S is closed & compact

$\therefore B(S)$ is.

b) note Since $B > 0$ $x \in T \Rightarrow x > 0$

$\therefore F$ is continous on T

c) Bolzano-Weierstrass Thm

Let $z \in T$ maximize $F(z) = \rho$.

Prob Find $y \in S$ that max F!

1) $z \in T$ then $y = \frac{z}{1^T z} \in S$

2) $F(y) = F(z) = \rho$

Claim $x \in S$ then $x' = (I+A)^{n-1} x \in T$

and $F(x') \geq F(x)$

to show $x \in S$ then $F(x) \leq F((I+A)x)$

ie $F(x) \geq \rho \Rightarrow F((I+A)x) \geq \rho$

$$F(x) \geq \rho \Rightarrow Ax \geq \rho x$$
$$\Rightarrow (I+A)Ax \geq (I+A)\rho x$$
$$\Leftrightarrow A(I+A)x \geq \rho(I+A)x$$
$$F((I+A)x) \geq \rho$$

note 1) $(I+A)A = A + A^2 = A(I+A)$

2) $B \geq 0$ & $x \geq y \geq 0$ then $Bx \geq By$

We have finished part a of Lemma 1

Def Support of a vector v, $\mathrm{supp}(v)$

$$\mathrm{supp}(v) = \{ i \mid v_i \neq 0 \}$$

Suppose part b) is false ie

$\exists x$ s.t $Ax \geq \rho x$ & $Ax \neq \rho x$

Claim $\exists y$ st $Ay \geq \rho' y$ $\rho' > \rho$

Pf

Def $T(x) = \{ i \mid (Ax)_i > \rho x_i \}$

Assume $T(x) \subsetneq \{1, \dots, n\}$

To show $\exists y$ $Ay \geq \rho y$ & $T(x) \subsetneq T(y)$

let $h \in \mathbb{R}^n$ s.t. $h_j = \begin{cases} 1 & \text{if } j \in T(x) \\ 0 & \text{o.w} \end{cases}$

set $y = x + \varepsilon h$ for small $\varepsilon > 0$

Consider

$$(Ay)_i - \rho y_i = (A(x + \varepsilon h))_i - \rho(x + \varepsilon h)_i$$

$$= ((Ax)_i - \rho x_i) + \varepsilon[(Ah)_i - \rho h_i]$$

Case 1 $i \in \tau(x)$ ie $(Ax)_i > \rho x_i$

for ε small $(Ay)_i > \rho(y)_i$

ie $i \in \tau(y)$

Case 2 $i \notin \tau(x)$ ie $(Ax)_i = \rho x_i$ & $h_i = 0$

$\therefore$ $(Ay)_i - \rho y_i = \varepsilon(Ah)_i \geq 0$

note A irred $\Rightarrow \exists\, i \notin \tau(x)$ st $(Ah)_i > 0$

$\tau(x) \subsetneq \tau(y)$ for ε small

$\therefore$ $\exists y$ st $\tau(y) = \{1, \ldots, n\}$ contra!

This proves Lemma 1 part b(i)

<u>pf</u> of Lemma 1 part b ii) i.e. $x > 0$

suppose $Ax = \rho x$ & $x_i = 0$

$\Rightarrow (Ax)_i = 0 = \sum_j A_{ij} x_j$

$\Rightarrow \forall j \; A_{ij} \neq 0$ that $x_j = 0$

i.e. $x_i = 0 \Rightarrow \text{Neig}(x_i) = 0$

Since G is strongly connected

$x = 0$ contra

<u>This proves Lemm 1</u>

Back to PF!

Lemma $A \geq 0$ & irred & ρ = max subharmonic A &
$|B| \leq A$ & $Bx = \theta x$ then

1) $|\theta| \leq \rho$

2) $|\theta| = \rho \Rightarrow |B| = A$ & $A|x| = \rho|x|$

Pf Suppose $Bx = \theta x$

$$|\theta||x| = |\theta x| = |Bx| \leq |B||x| \leq A|x| \qquad (*)$$

$\therefore$ $|x|$ $|\theta|$-subharmonic for A

Thus $|\theta| \leq \rho$

Suppose $|\theta| = \rho$ by $(*)$

$$A|x| = |B||x| = \rho|x|$$

by previous thm $|x| > 0$

$$\therefore \quad A - |B| \geq 0 \ \& \ (A - |B|)|x| = 0$$

$$\Rightarrow A - |B| = 0$$

$$A = |B|$$

Eigenvalue Multiplicity

Algebraic & Geometric

eg $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ $\quad P(\lambda) = \det\begin{pmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{pmatrix} = (1-\lambda)^2$

$\lambda = (1,1)$ Alg Mult $= 2$

Eigenvectors

$$\text{Null}\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = \text{Null}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle$$

Geo Mult 1

Thm $\rho = \rho(A)$ & $A \geq 0$ & irred then

1) Geom Mult of ρ is 1

2) $x \in \rho$-eigen space then $\text{sign}(x_i) = \text{sign}(x_j)$

3) Algebraic Mult of ρ 1

---

pf

1) Suppose $x, y \in \rho$-eigen space & x, y ind

$\exists\, x + \alpha y$ not of full support! contra!

2) $Ax = \rho x \Rightarrow A|x| = \rho|x|$ by previous lemma

$\Rightarrow \text{sign}(x_i) = \text{sign}(x_j)$

3) Let $K = \mathrm{Null}(A - \rho I)$ ie $x \in K$ iff $(A-\rho I)x = 0$

$C = \mathrm{Col}(A - \rho I)$

$\dim(K) = 1$ and thus $\dim(C) = n-1$

<u>Claim</u> $K \cap C = 0$

Suppose $x \in K \cap C$

1) $x \in C \Rightarrow \exists y \quad x = (A-\rho I)y \qquad (*)$

2) $x \in K \Rightarrow$ WLOG $x > 0$ (Thm)

<u>note</u> $(A-\rho I)(y + kx) = x$

thus for k large assume $y > 0$

rewrite $(*)$ $\quad Ay = \rho y + x \Rightarrow Ay \geq \rho y$

y is ρ-subharmonic $\Rightarrow y \in K \Rightarrow x = 0$ Done.

<u>Claim</u> $\mathrm{char}(A, t) = (t - \rho)\, \mathrm{char}(A\restriction C, t)$

$\therefore$ Alg Mult $= 1$