

Solving Linear Systems

15-859N
9/26/11

M is $n \times n$ sparse matrix over some field.

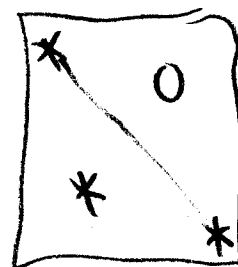
$$w = \# \text{ nonzero elements} = O(n) \quad w = w(M)$$

Goal: given M, b solve $MX = b$

- 1) Direct Methods
- 2) Iterative Methods
- 3) Hybrid Methods

Direct Methods

Suppose L is lower triangular and $L =$



Claim $LX = b$ can be solved in $O(w(L))$
by back substitution

Pf $l_{11}x_1 = b_1 \Rightarrow x_1 = b_1/l_1$ if $l_1 \neq 0$

$$\text{solve } \begin{pmatrix} l_{22} & 0 \\ * & l_{nn} \end{pmatrix} \begin{pmatrix} x_2 \\ \vdots \\ x_n \end{pmatrix} = b'$$

Suppose $M = LU$ L lower tri
 U upper tri

Then to solve $LUx = b$ in $O(w(L) + w(U))$ time.

Alg solve $LY = b$
 solve $UX = Y$

Computing an LU decomposition
 (Cholesky Decomposition)
 Gaussian Elimination - Pivoting

Suppose $M = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix}$

Subtract row 1 from 2 & 3

$$\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

Goal: subtract row 2 from row 3

Stuck!

1) Swap rows 2 & 3

2) start over

$$\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

In General (Assume we have pre ordered rows)
($M \leftarrow PM$)

$$\text{Let } L_i = \begin{pmatrix} 1 & & & 0 \\ & \ddots & & \\ 0 & * & \ddots & \\ & & & 1 \end{pmatrix}$$

i

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ & * & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ 0 & 1 & \\ & * & 1 \end{pmatrix} \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & * & 1 \end{pmatrix} M = U$$

$$\text{ie } L_{n-1} \cdots L_2 L_1 M = U$$

note $L_{n-1} \cdots L_1 \equiv \text{lower tri}$

The product is a mess

Consider $L^{-1} = L_1^{-1} \cdots L_{n-1}^{-1}$

Claim: if $L_1 = \begin{pmatrix} 1 & & & \\ a_1 & 1 & & \\ \vdots & & \ddots & \\ a_{n-1} & & & 1 \end{pmatrix}$ then $L_1^{-1} = \begin{pmatrix} 1 & & & \\ -a_1 & 1 & & 0 \\ \vdots & & \ddots & \\ -a_{n-1} & & & 1 \end{pmatrix}$

if $L_2 = \begin{pmatrix} 1 & & & \\ & 1 & & 0 \\ 0 & b_1 & & \\ \vdots & & \ddots & \\ 0 & b_{n-2} & & 1 \end{pmatrix}$

then $L_1^{-1} L_2^{-1} = \begin{pmatrix} 1 & & & \\ -a_1 & 1 & & 0 \\ \vdots & & \ddots & \\ -a_{n-1} & -b_{n-2} & 0 & 1 \end{pmatrix}$

Claim $L^{-1} = nI - L_1 - \cdots - L_{n-1}$

$$M = L^{-1}u$$

When can we pick any pivot order?

Answer Symmetric positive definite (important case)

ie 1) Symmetric $M^T = M$

2) Positive definite ie $x \neq 0 \Rightarrow x^T A x > 0$

ie eigenvalues of $A > 0$

Ex If M is spd then $\text{Diagonal}(M) > 0$
and stays positive under Gaussian elimination.

Goal: Pick pivot order to minimize $(w(L))$
for spd matrices

Def Graph of $M^{n \times n} \equiv (V, E) = G$

$$V = \{v_1, \dots, v_n\}$$

$$(v_i, v_j) \in E \text{ if } M_{ij} \neq 0$$

G is undirected

Schur complement

Suppose $L_1 M = \begin{pmatrix} a & * & - & * \\ 0 & \boxed{M_1} \\ \vdots & \\ 0 \end{pmatrix}$

$$L_1 M L_1^T = \begin{pmatrix} a & 0 & - & 0 \\ 0 & \boxed{M_1} \\ \vdots & \\ 0 \end{pmatrix}$$

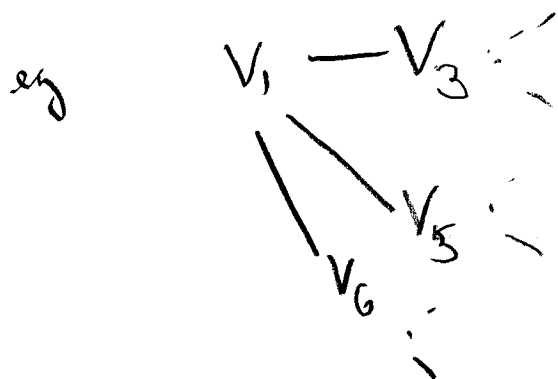
$$(L_1 M L_1^T)^T = (L_1^T)^T M^T L_1^T = L_1 M L_1^T$$

$$\Rightarrow M_1^T = M_1$$

Def $M_1 \equiv$ Schur Complement

Question $\text{Graph}(M_1)?$

$G = \text{Graph}(M)$



$$M = \begin{pmatrix} & 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & * & 0 & * & 0 & * & * & 0 \\ 2 & 0 & * & 0 & ? & & & \\ 3 & * & & & & & & \\ 4 & 0 & & & & & & \\ 5 & * & & & & & & \\ 6 & * & & & & & & \end{pmatrix}$$

$$L, M = \begin{pmatrix} & 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & * & 0 & * & 0 & * & * \\ 2 & 0 & & & & & \\ 3 & 0 & & * & * & * & \\ 4 & 0 & & & & & \\ 5 & 0 & & * & * & * & \\ 6 & 0 & & * & * & * & \end{pmatrix}$$

possibly
new edges
in M'

$E(3,5)$ new $E(3,1) \& E(1,5)$
 $E(3,6)$ new $E(3,1) \& E(1,6)$

Graph theoretic view of pivoting

$\text{pivot}(V_i) \equiv$ 1) Make a clique of $\text{Neig}(V_i)$
2) remove V_i and its edges

Fill $\equiv$ edges added at step 1

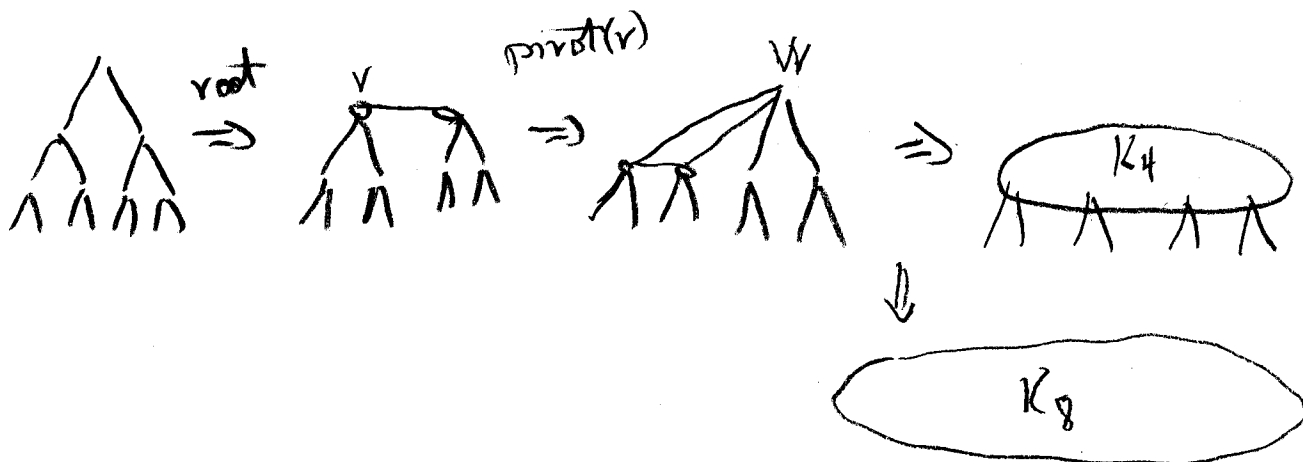
Cost of $\text{pivot}(V_i)$ $\equiv O(d_i^2)$ where

$d_i^2 \equiv$ degree of V_i at pivot-time.

Pivot Strategy

EG $G \equiv$ balanced binary tree

1) Consider pivoting from root to leaves



Fill $\equiv \Omega(n^2)$ we get clique on leaves
 $K_{n/2}$

Work the work to pivot $K_{n/2}$ on leaves
 $\Omega(n^3)$

2) Consider pivoting: leaves to root

note All pivots are on degree one nodes!

✓ 1) no fill

✓ 2) $O(n)$ work



Perfect Elimination

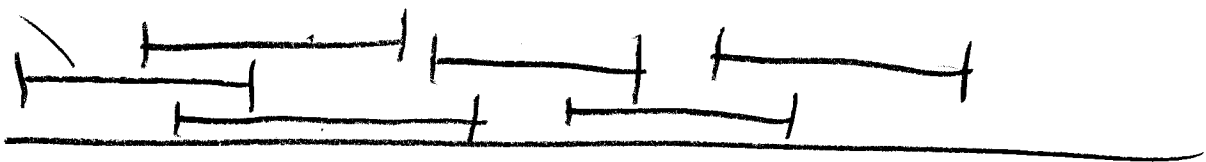
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Def A perfect elimination order is one with no fill.

Note Trees have perfect elim orders

Def G is an interval graph if

- 1) each vertex corresponds to an interval
- 2) $(v_i, v_j) \in E$ if $I_i \cap I_j \neq \emptyset$

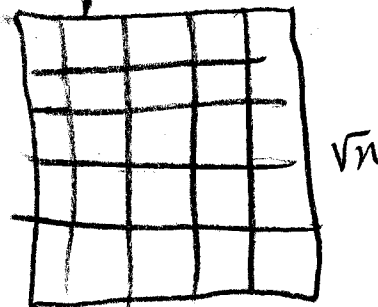


Fact Interval graph has a perfect elim order.

Note If I has left most right end point then $\text{Neig}(I)$ is a clique.

Pivot Strategies for Sq Mesh

Input $AX=b$ where $A^{n \times n} =$

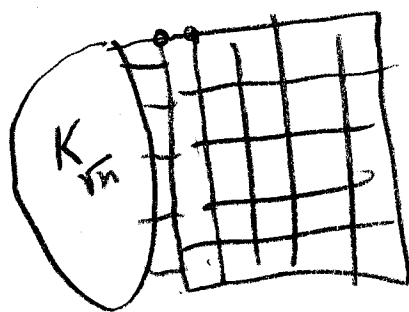


Known: $\text{Fill}(\text{Mesh}) = \Theta(n \log n)$

$\text{Work}(\text{Mesh}) = \Theta(n^{3/2})$

Try: One column at a time

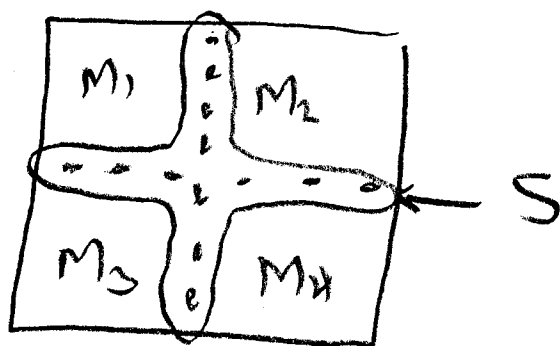
After first column



Each column will be a clique $\Omega(n^{3/2})$ fill

$\Omega(n^2)$ work.

Try Nested Dissection



$$\text{Pivot}(M) \equiv (\text{Pivot}(M_1), \text{Pivot}(M_2), \text{Pivot}(M_3), \\ \text{Pivot}(M_4), S)$$

Goal: Bd Fill & work for ND!

View of Fill for Planar Graphs

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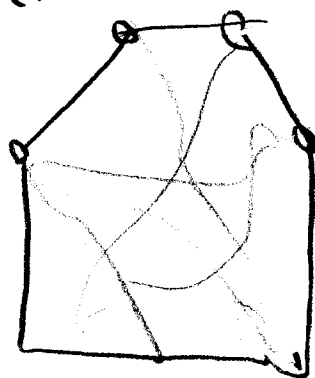
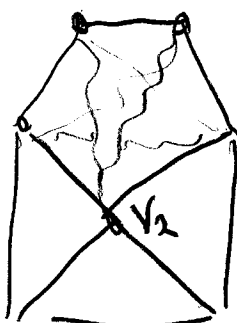
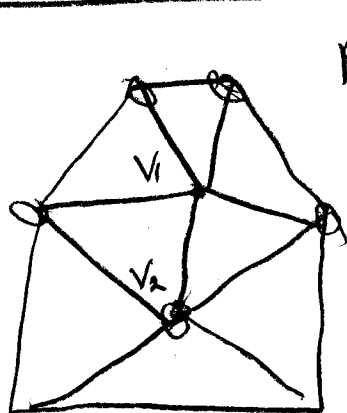
Input: 1) Triangulated planar graph G
2) Pivot strategy $v_1, \dots, v_n$

Two Views: 1) $G_i \equiv G \setminus v_1, \dots, v_i$

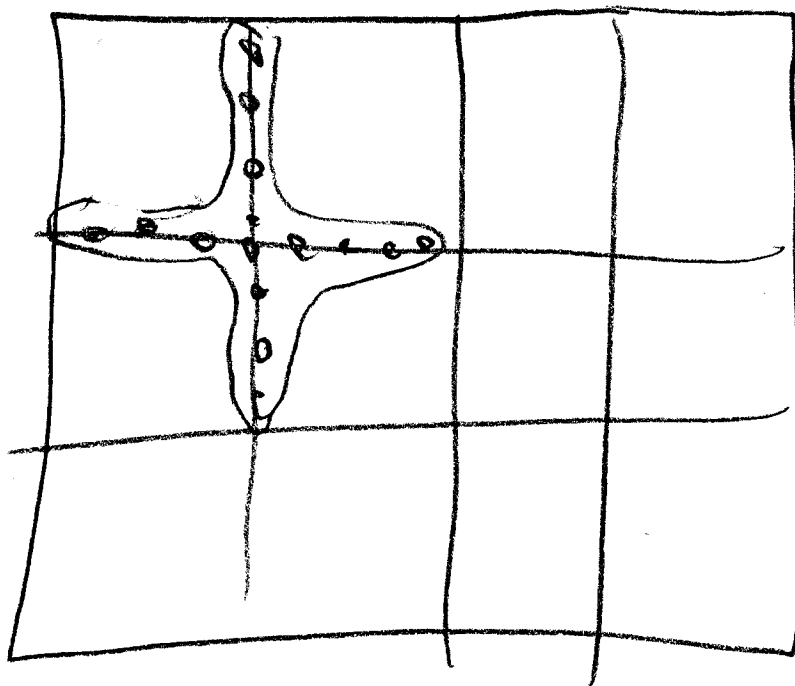
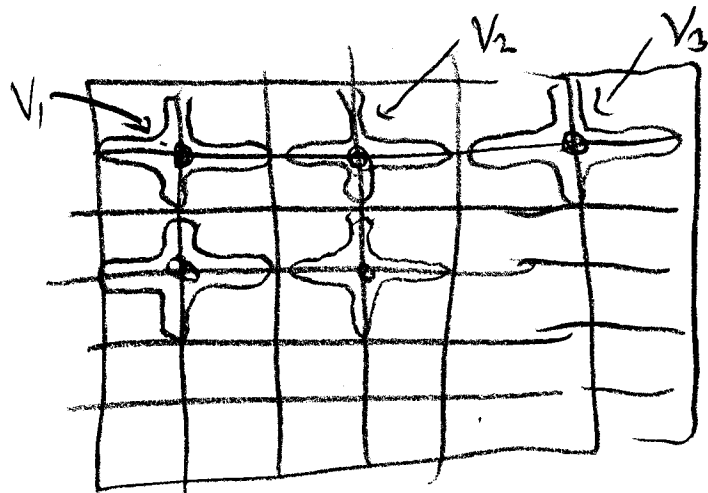
2) $F_i \equiv G$ after pivot $v_1, \dots, v_i$

View of elimination - Pivot (v_i)

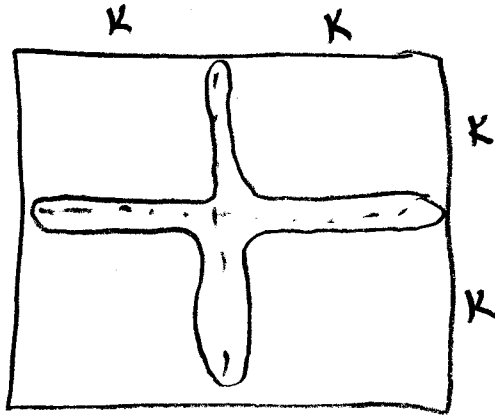
Given G_{i-1} 1) remove v_i from G_{i-1} forming new faces
2) Form a clique out of vertices on a face.



Analysis of pivoting on crosses for Sg mesh



Removing Crosses of Size $2K \times 2K$



View pivot of a cross in 2 steps

- 1) Pivot horizontal part
- 2) Pivot out vertical part.

Computing the Fill for some K

$$\# \text{ crosses } \left(\frac{\sqrt{n}}{2K} \right)^2 = \frac{n}{4K^2}$$

- 1) Initially the faces are size $4K$.

The first pivot will form a new face from two faces

This fill = $(3K)^2$ and $\frac{n}{2K^2}$ faces

The remaining pivots have no fill

$$\text{Fill}_H = \left(\frac{n}{2K^2} \right) (3K)^2 = \frac{9}{2}n$$

2) Initially faces have size $6K$.

Only the first pivot per vertical part has fill.

$$\text{Fill}_V = \left(\frac{n}{4K^2} \right) (4K)^2 = 4n$$

∴ Total fill $\sum_{i=1}^{\log \sqrt{n}} \frac{17}{2} n \approx \frac{17}{2} n \log n = O(n \log n)$

The Work for ND

$$1) (\# \text{ crosses}) (\# \text{ pivots}) \frac{(\text{face size})^3}{2}$$

$$\left(\frac{n}{4k^2}\right) (2k) \left(\frac{(2k)^2}{2}\right) \leq \frac{49}{4} nk = cnk$$

$$2) \left(\frac{n}{4k^2}\right) (2k) \left(\frac{(16k)^2}{2}\right) = cnk$$

$$\text{Total Work} \left(\sum_{i=1}^{\log \sqrt{n}} cn 2^i \right) = O(n^{3/2})$$

$$k = 2^i$$

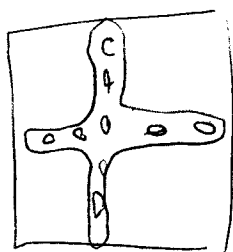
Block Gaussian Elimination

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$$\begin{pmatrix} \pm 0 \\ -BA^{-1}I \end{pmatrix} \begin{pmatrix} A & B^T \\ B & C \end{pmatrix} = \begin{pmatrix} A & B^T \\ 0 & C - BA^{-1}B^T \end{pmatrix}$$

Schur complement $D = C - BA^{-1}B^T$

Pivot-out crosses



$A^{4K \times 4K}$

B has only $8K$ nonzero rows

Alg	work	time
1) compute A^{-1}	$O(K^2)$	$O(1/3 K)$
2) BA^{-1}	$O(K^2)$	$O(1/3 K)$
3) $(BA^{-1})B^T$	$O(K^2)$	$O(1/3 K)$
	$O(K^2)$	$O(1/3 K)$ per cross

Work per level

$$\left(\frac{n}{4k^2}\right) c k^2 = c k^{\alpha-2} n$$

$$k = 2^L$$

$$\text{Total Work } c n \sum_{i=1}^{\log n} (2^i)^{\alpha-2} = c n (\sqrt{n})^{\alpha-2} = c n^{\alpha/2}$$