

More Direct Methods

15-859N

9/28/11

$$G = (V, E)$$

Def $S \subseteq V$ is an $f(n)$ -separator if (vertex)

1) $|S| \leq f(n)$

2) $\exists$ partition A, B of $V - S$

3) $|A|, |B| \leq \frac{2}{3}n$

Gilbert-Tarjan $S \subseteq V$ is an (α, β) - $f(n)$ -separator

$\alpha < 1$ & $\beta > 0$

1) $|S| \leq \beta f(n)$

2) $\exists$ part A, B of $V - S$

3) $|A|$ & $|B| \leq \alpha n$

Lipton-Tarjan If G is planar $\exists \sqrt{n}$ -separator

$\alpha = \frac{2}{3}, \beta = \sqrt{4}$

Conjecture $\beta = \sqrt{n}$ works.

Pivot Strategies

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1) (Nested Dissection) (ND)

- 1) Find a vertex separator S of G .
- 2) Let $G_1, \dots, G_k$ be connected components of $G-S$
- 3) Return $ND(G_1), \dots, ND(G_k), S$

2) (Lipton-Rose-Tarjan) (Generalized ND)

Procedure $LRT(G, H \subseteq V)$

* Returns an ordering of $V-H$

* initially $H = \emptyset$

- 1) Find vertex separator S of G & parts $A, B \subseteq V$

2) Let

$$G_1 = (A \cup S, E(A \cup S) - E(S)) \quad H_1 = S \cup (H \cap A)$$

$$G_2 = (B \cup S, E(B \cup S) - E(S)) \quad H_2 = S \cup (H \cap B)$$

3) return

$$LRT(G_1, H_1), LRT(G_2, H_2), S-H$$

Fill and Work Results

$\mathcal{G}_\alpha$ be a class of graphs closed under subgraphs
s.t.

1) Each graph has an n^α -separator.

Thm (LRT) For $G \in \mathcal{G}_{1/2}$

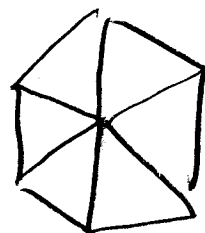
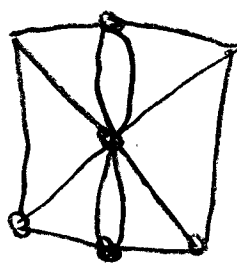
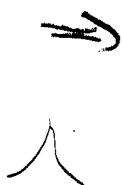
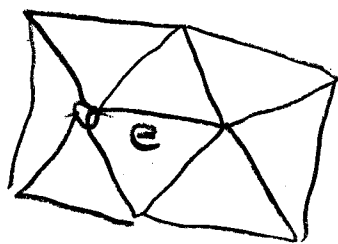
GND gives $O(n \log n)$ fill

$O(n^{3/2})$ work.

Gilbert-Tarjan

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Graph Contraction



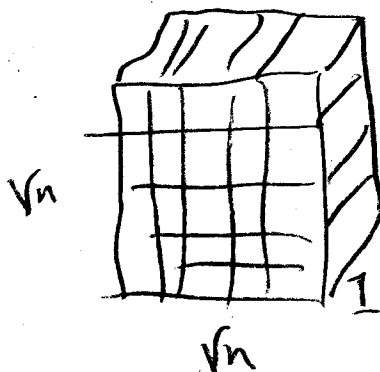
edge contraction

remove mult edges

Def G is sparse-contractible if every m -node contraction of G has $O(m)$ edges.

Note Planar graphs are sparse-contractible

Note $P_m \times P_m \times P_1$ is not sparse-contractible



It has a K_m minor.

Example Let P_n be path graph
 S_n be star graph

$$G_k = P_k \otimes S_k$$

Claim All subgraphs of G_k have $\sqrt{n}$ -separators
 pt in GT

Claim $\exists$ ND ordering for G_k with fill $\Omega(n^{5/4})$
 $n = k^2$

order

1) Top level sep $\equiv$ "spine" of G_k

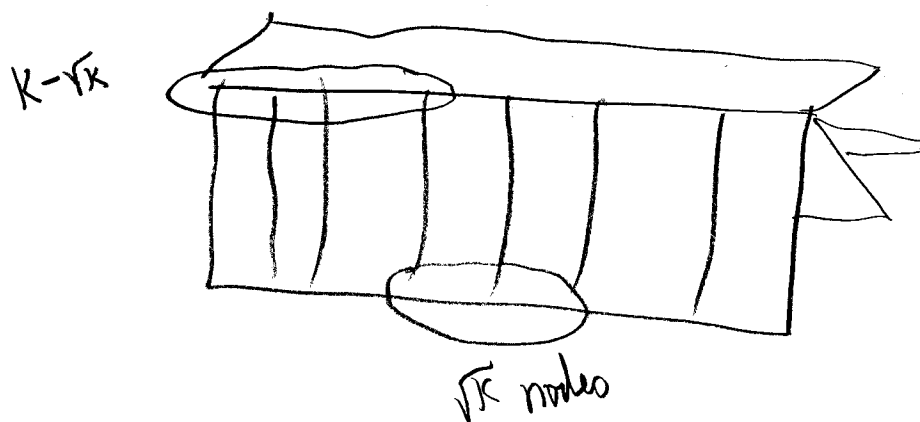
This gives k paths after removing spine.

2) For each path P_k pick middle $\sqrt{k}$ nodes.

3) For each middle $\sqrt{k}$ nodes pivot "outside in".

4) Pick remaining node any way you like

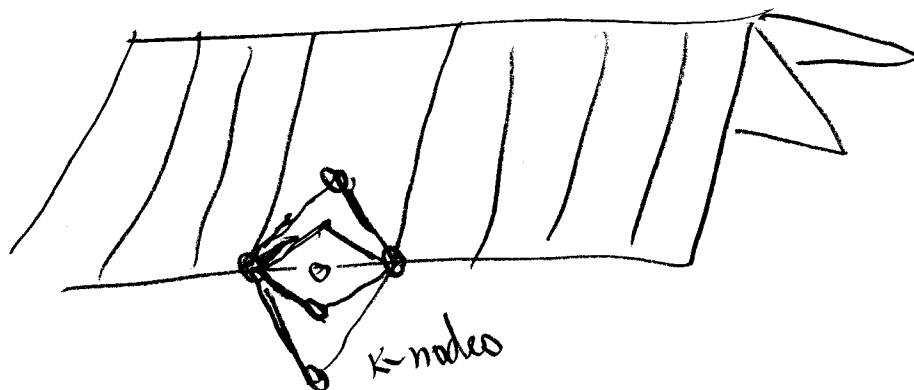
Consider one wing from the spine



As we pivot the $\sqrt{K}$ nodes we get $(K - \sqrt{K})(\sqrt{K}/2)$ fill
per wing $\Omega(K^{3/2})$ fill

Over all wings $\Omega(K^{5/2}) = \Omega(n^{3/4})$ fill.

Minimal separator version



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Min Degree Heuristic Fractals

MDH $\equiv$ Pivot on a variable with minimum degree.

Question How do we break ties?

Consider MDH in planar case.

Simple fill model for planar graphs

1) Start with Δ planar graph (genus g surface)

2) Pivot $\equiv$ remove a vertex & its edges

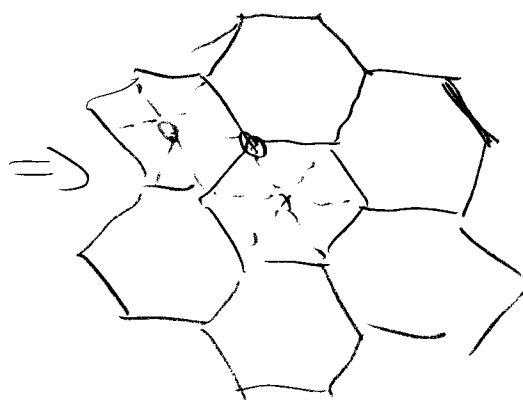
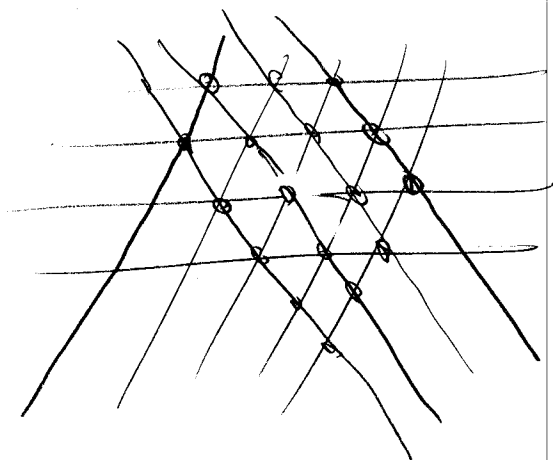
3) Fill at a given time

$\{(x, y) \mid x \& y \text{ share a face}\}$

Fractals

One way to generate them.

- 1) Start with infinite 2d plane
- 2) remove a "periodic" subset of vertices



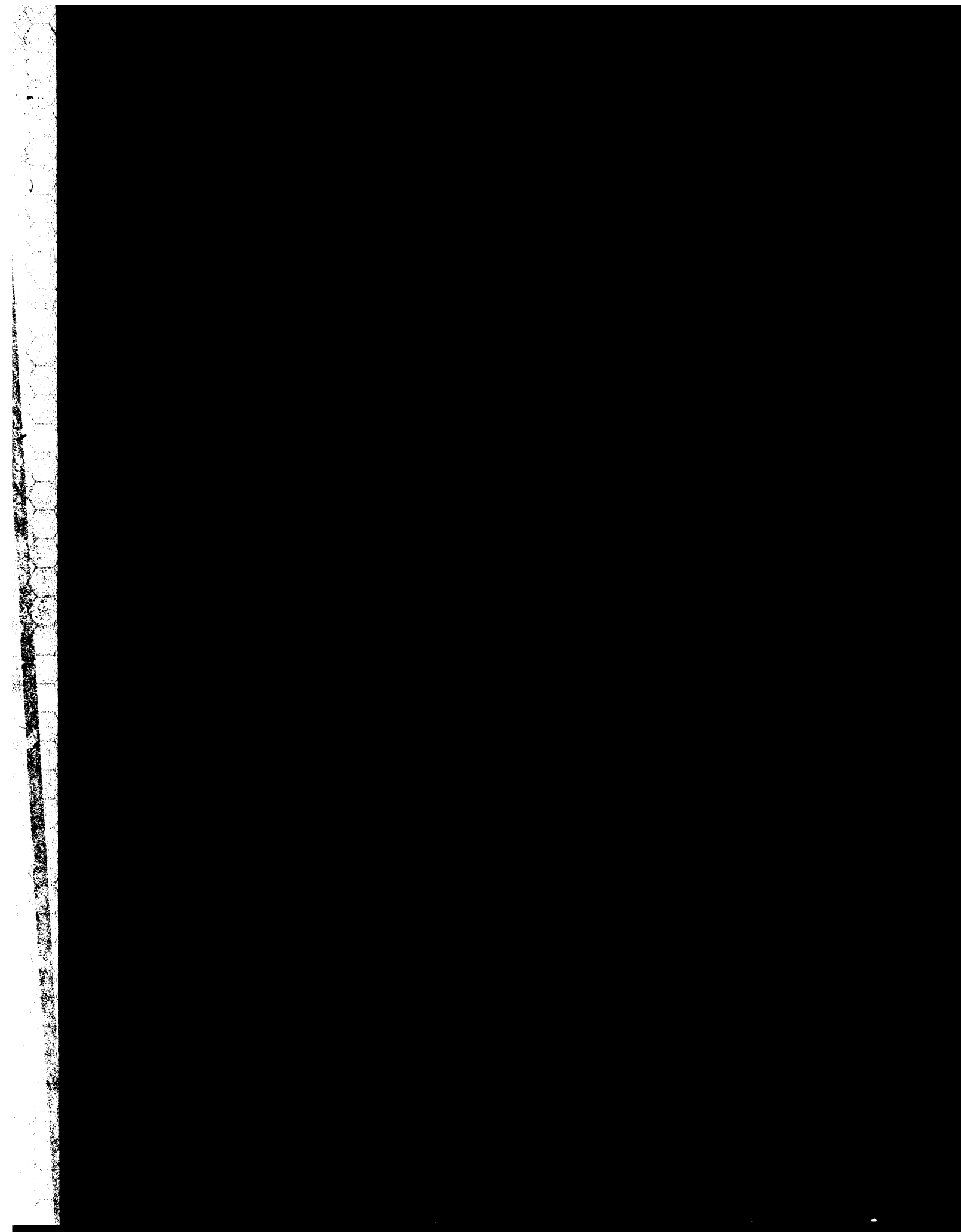
Main Issue what is area to bdry?

For an $N \times N$ sq $A = N^2$ $B \approx N$

$$\dim \text{bdry} = \frac{\log(\text{length})^2}{\log(\text{area})} = \frac{\log N^2}{\log N^2} = 1$$

First Alex case $A_{i+1} = 7A_i$ & $B_{i+1} = 3B_i$

$$\dim \text{bdry} = \frac{\log(3^k)^2}{\log(7^k)} = \left(\frac{1 \log 9}{1 \log 7} \right) = \frac{\log 9}{\log 7} = 1.16$$

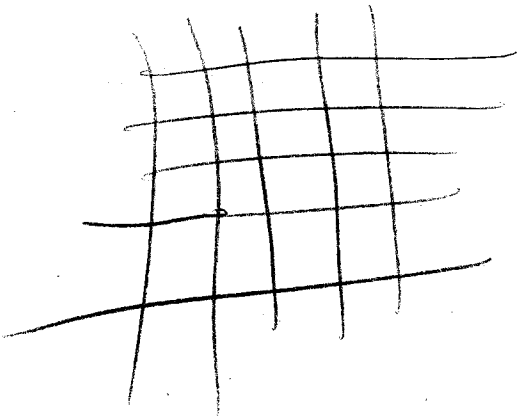




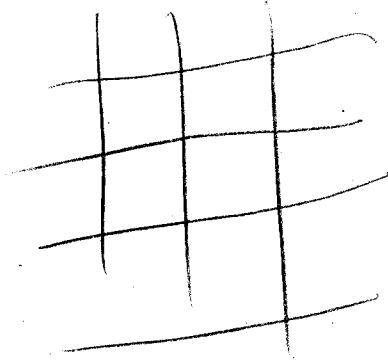




Fill in linear case



halb
 $\Rightarrow$



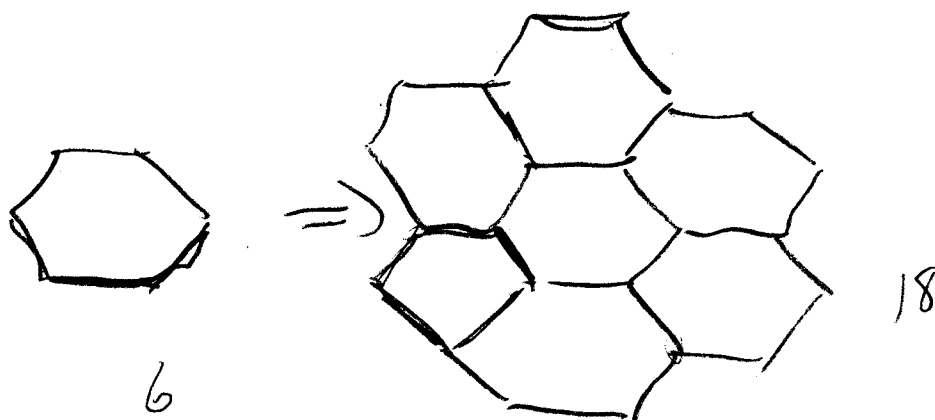
$$A_{i+1} = 4A_i \Rightarrow \text{\# rounds } \log_4 n$$

$$B_{i+1} = 2B_i$$

Find bdy size $2^{\log_4 n} = n^{\log_4 2} = \sqrt{n}$

$$\text{fill} \geq (\sqrt{n})^2 = n$$

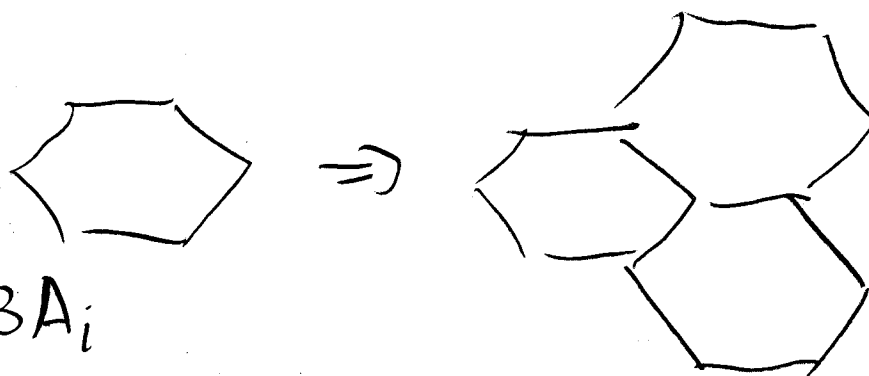
Fill for Hex example



$$A_{i+1} = 7A_i \quad \# \text{ rounds} = \log_7 n$$

$$B_{i+1} = 3B_i \quad \text{bdary size} = 3^{\log_7 n}$$

$$\text{fill} = \left(3^{\log_7 n}\right)^2 = \left(n^{\log_7 3}\right)^2 = n^{\log_7 9} \approx n^{1.12}$$



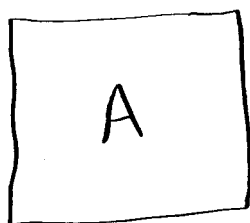
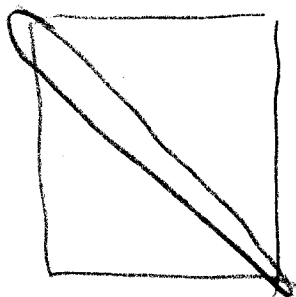
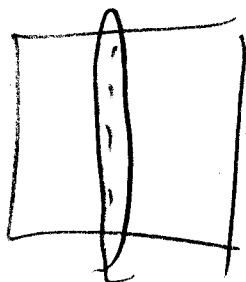
$$A_{i+1} = 3A_i$$

$$B_{i+1} = 2B_i$$

$$\text{fill} = \left(n^{\log_3 2}\right)^2 = n^{\log_3 4} \approx n^{1.26}$$

Diagonal cuts for 2D-Mesh

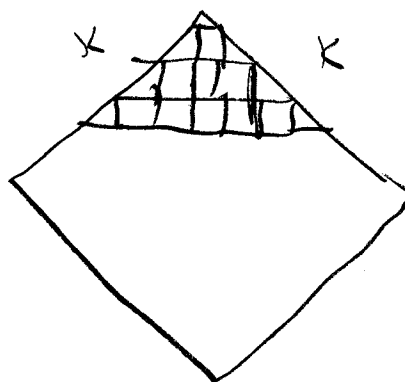
Vertical



$$B = 4K$$

$$\text{Fill} = 16K^2$$

$$\text{Area} = K^2$$



$$A = (\sqrt{2} K)^2 = 2K^2$$

$$B = 4K$$

normalize

$$A = K^2 = (\sqrt{2} (K/\sqrt{2}))^2$$

$$B = 4(K/\sqrt{2}) = 2\sqrt{2}K$$

$$\text{Fill} = (2\sqrt{2}K)^2 = 8 \cdot K^2$$

2-time better