15-859N
11/02/11

Fiedler's Thm

Generalized Laplacians

Throughout $G=(V,E)$ unweighted & undirected

Def $Q^{n \times n}$ is a generalized Laplacian

if:
1) Real symmetric
2) $Q_{ij} < 0$ if $(i,j) \in E$
3) $Q_{ij} = 0$ if $i \neq j$ & $(i,j) \notin E$
4) Q_{ii} no constraint

Note $\lambda(Q)$ are real say

$\lambda_1 \cdots \lambda_n$

$x_1 \quad x_n$

When needed we can set $\lambda_2 = 0$

ie $(Q - \lambda_2 I)$

<u>Lemma</u> A G connected then $\lambda_1(Q)$ is simple and eigenvector x_1 can be picked $x_1 > 0$.

<u>pf</u> Pick c s.t. $(-Q+cI) \geq 0$ ie

Digonal $(Q-cI) \leq 0$

By Perron-Frobenius $\lambda_n(-Q+cI)$ is simple & can be picked positive.

<u>Def</u> $x: V \to \mathbb{R}$ is an <u>valuation</u>.

<u>Positive support</u> of $x = \{v \in V \mid x(v) > 0\} = Supp_+(x)$.

<u>Negative</u> " $\quad$ " $< 0\} = Supp_-(x)$.

Def Positive nodal domain $\equiv$

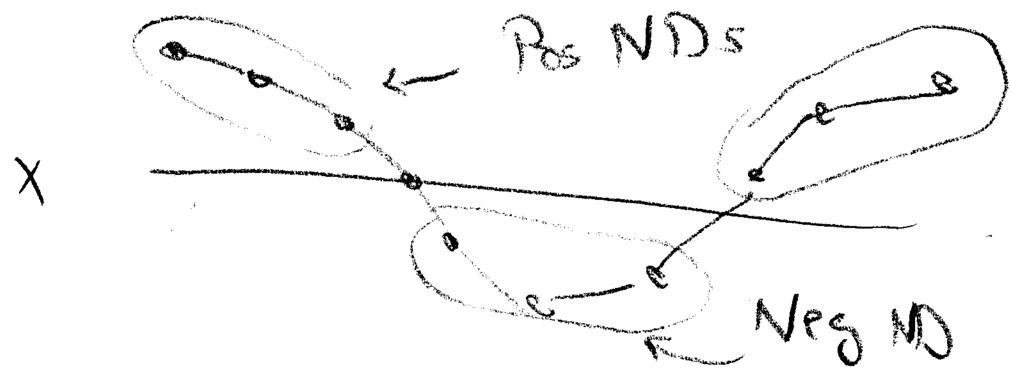
Connected component of induced subgraph on $\mathrm{Supp}_+(x)$

Negative ND $\equiv$

Nodal Domain $\equiv$ Positive or Neg ND.

Goal: Better understand ND.

EG Valuation x for P_n



Def Y is a ND of x.

$$(x_Y)_i = \begin{cases} |x_i| & \text{if } i \in Y \\ 0 & \text{o.w.} \end{cases}$$

<u>Lemma</u> $Qx \leq \lambda x \wedge$ Pos ND Y of x then

$$(Q - \lambda I) x_Y \leq 0 \quad \text{ie} \quad Q x_Y \leq \lambda x_Y$$

<u>pf</u>

Write $x = \begin{pmatrix} y \\ \bar{y} \\ z \end{pmatrix}$ $\begin{matrix} \leftarrow V(Y) \\ \leftarrow \text{other Pos NDs} \\ \leftarrow \text{Neg NDs \& zeros} \end{matrix}$

Write Q as

$$\left(\begin{array}{c|c|c} Q_Y & A_Y & B_Y \\ \hline A_Y^T & & \\ \hline B_Y^T & & \end{array}\right) \begin{pmatrix} y \\ \bar{y} \\ z \end{pmatrix} \leq \lambda \begin{pmatrix} y \\ \bar{y} \\ z \end{pmatrix}$$

<u>Note</u> $A_Y \equiv 0$ & $B_Y \leq 0$ & $z \leq 0$

Thus $Q_Y y + B_Y z \leq \lambda y$

since $B_y z \geq 0 \Rightarrow Q_Y y \leq \lambda y$

$A_y^T = 0 \Rightarrow A_Y^T y = 0$

$B_Y^T \leq 0 \Rightarrow B_Y^T y \leq 0$

Thus $Q \begin{pmatrix} y \\ 0 \\ 0 \end{pmatrix} \leq \lambda \begin{pmatrix} y \\ 0 \\ 0 \end{pmatrix}$

Cor $Qx \leq \lambda x$ & $\mathcal{U} = \langle x_Y \mid Y \text{ Pos ND of } x \rangle$

$u \in \mathcal{U} \Rightarrow u^T (Q - \lambda I) u \leq 0$

pf $u \in \mathcal{U} \Rightarrow u = \sum_{Y \in PND} \alpha_Y x_Y$

note $Y \neq Y' \Rightarrow x_Y^T Q x_{Y'} = 0$ & $x_Y^T x_Y \leq 0$

$u^T (Q - \lambda I) u = \sum_{Y \in PND} \alpha_Y^2 x_Y^T (Q - \lambda I) x_Y \leq 0$

<u>Def</u> $\text{Support}(x) = \{v \mid x(v) \neq 0\}$

<u>If</u> $Ax = \lambda x$ then x has <u>minimal support</u>
if $\forall y$ s.t. $Ay = \lambda y \Rightarrow \text{Support}(y) \not\subsetneq \text{Support}(x)$.

<u>Main Thm</u> Q gen Lap of G, G connected, $Qx = \lambda_2 x$, and x has minmal support then $\text{supp}_+(x)$ & $\text{supp}_-(x)$ induce connected subgraphs.

Pf by contradiction

Suppose $Qv = \lambda_2 v \wedge$ PND Y & Z of v.

Lemma A $\Rightarrow \exists x > 0$ st $Qx = \lambda_1 x$.

$v_Y^T v_Z = 0 \Rightarrow \dim \langle v_Y, v_Z \rangle = 2$

$\Rightarrow \exists u \in \langle v_Y, v_Z \rangle$ st $u^T x = 0$

But $u^T x = 0 \Rightarrow \dfrac{u^T Q u}{u^T u} \geq \lambda_2$ ie $u^T(Q - \lambda_2 I)u \geq 0$

By Cor $u^T(Q - \lambda_2 I)u \leq 0$

Thus $u^T(Q - \lambda_2 I)x = 0$

By Corrant-Fischer

$u^T x = 0$ & $\dfrac{u^T Q u}{u^T u} = \lambda_2 \Rightarrow Qu = \lambda_2 u$

$\text{Support}(u) \subseteq V(Y) \cup V(Z) \subsetneq \text{Support}(v)$

But v was minimal! contra!

<u>Def</u> Going from v to u we call the <u>Fiedler Flip</u>.

Question: Can we add Fiedler Flip to an iterative algorithm without increasing $\dfrac{v^T Q v}{v^T v}$?

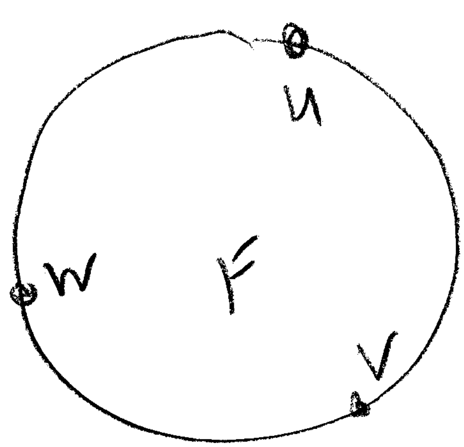
Applications of ND thm.

Lemma G is a 3-connected embedded planar,
$Qx = \lambda_2 x$ then no 3 vertices of a face are zero.

Pf by contradiction.

We give idea first:

Suppose 1) $x(u) = x(v) = x(w) = 0$
2) x has minimal support
3) $p \in Supp_+(x)$ & $q \in Supp_-(x)$

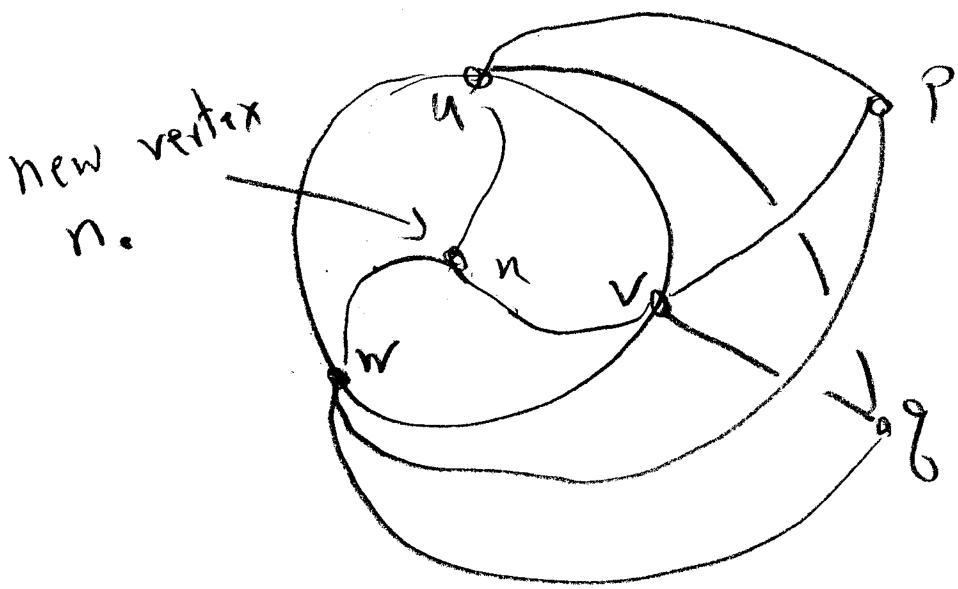


By Menger's Thm

$\exists$ 3 vertex disjoint paths from u, v, w to p

" " q

If paths to p & to q are also disjoint then



This is $K_{3,3}$ contra!.

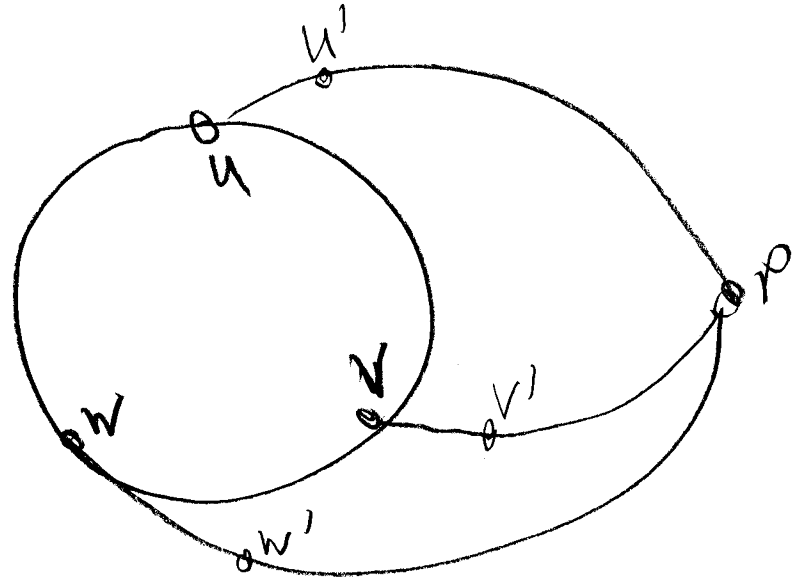
Full pf

Note if $x(v) = 0$ & $\exists$ Neig$(v) = v'$ & $x(v') > 0$

then $\exists$ Neig$(v) = v''$ s.t $x(v'') < 0$

pf $Qx = \lambda x \Rightarrow x(v)$ convex comb of neigs.

consider vertex disj paths to p



Let u' be first $u' \in$ Path(u,p) st

$x(u') = 0$ & $x($Next$(u')) \neq 0$

P_u = path from u to u'. Sim define P_v, P_w.

Contract $Supp_+(x)$, $Supp_-(x)$, P_u, P_v, P_w

this gives $K_{3,2}$ contra!

<u>Cor</u> G 3-con & planar $\Rightarrow \lambda_2(Q)$ has multiplicity ≤ 3.

<u>pf</u> by contra!

Suppose 1) mult ≥ 4 with ind vectors x_1, x_2, x_3, x_4.

2) Vertices u, v, w on face F.

<u>Claim</u> $\exists x \in \lambda_2(Q)$, $x \neq 0$, zero on u, v, w.

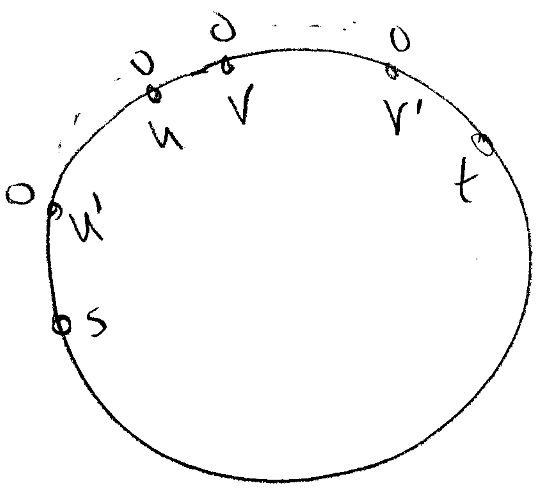
$$\begin{array}{c c} & \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{array}{c} u \\ v \\ w \end{array} & \left[\begin{array}{c|c|c|c} a_1 & a_2 & & a_4 \\ \hline b_1 & & & \\ \hline c_1 & & & \end{array}\right] \end{array} \begin{pmatrix} \alpha_1 \\ \\ \\ \alpha_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\exists \alpha \neq 0$ contra!

<u>Lemma</u> G is 2-con embedded planar graph $\wedge$
$Qx = \lambda_2 x$ $\wedge$ x of minimal support if $(u,v) \in E$ of
Face F s.t $x(u) = x(v) = 0$ Then $x(F) \geq 0$ or $x(F) \leq 0$.

pf G 2-con $\Rightarrow$ F is a simple cycle

Assume $(u,v) \in E \subseteq F$ & $x(u) = x(v) = 0$
and $\exists p, q \in F$ s.t $x(p) > 0$ & $x(q) < 0$

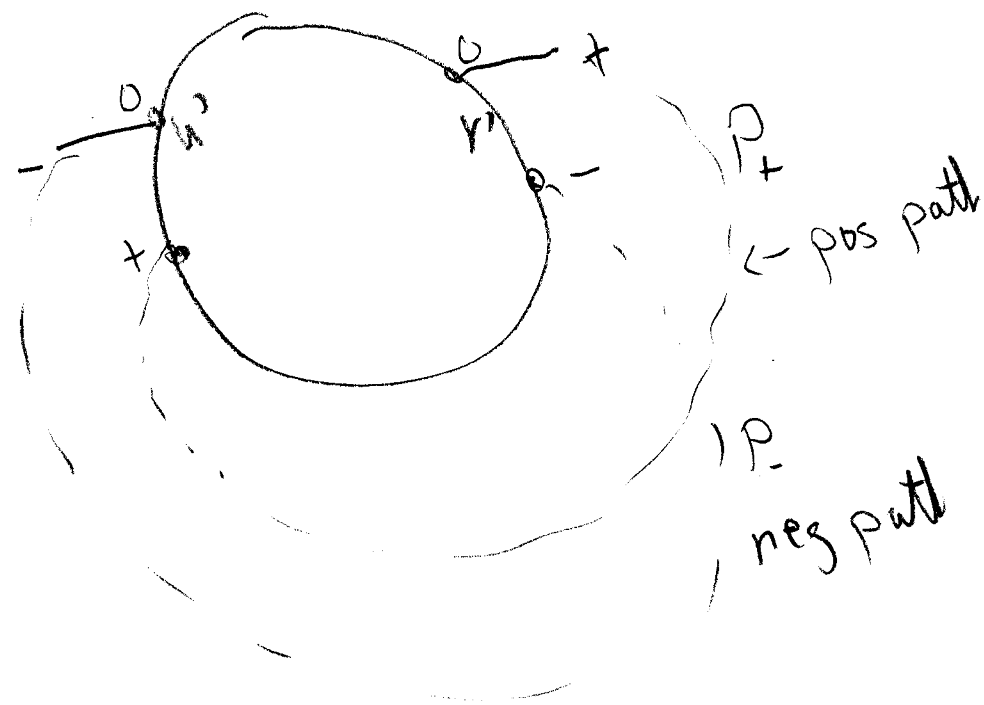


Interval $\langle u', v' \rangle$ is a maximal zero interval of x

u', v' have both pos & neg neigh.

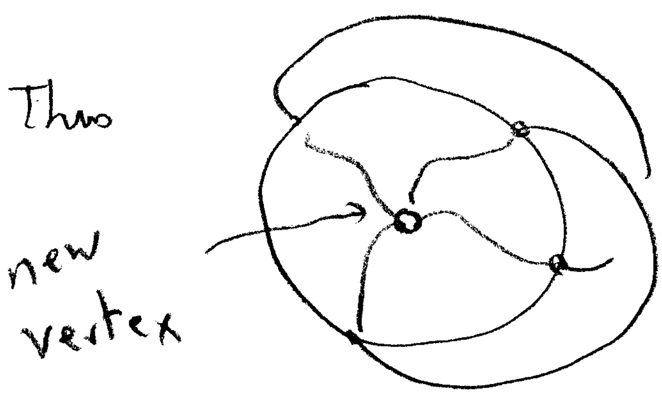
Case 1 $x(s) > 0$ & $x(t) < 0$

Case 1



<u>Claim</u> $\exists$ subpaths of P_+, P_- disjoint from F that "interlace".

Thus

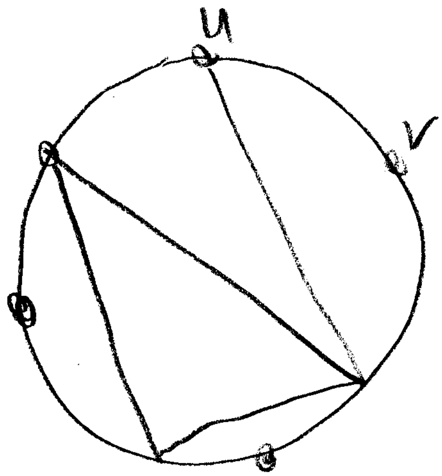


This is K_5 a contra!

Case 2 $\quad x(s)<0$ & $x(t)>0$ $\quad$ similiar

<u>Cor</u> G is 2-conn & outerplanar then
$mult(\lambda_2(Q)) \leq 2$.

<u>pf</u> by contra.



assuming mult ≥ 3 then $\exists x \in \lambda_2(Q)$

st $x(u) = x(v) = 0$

$\Rightarrow x \geq 0$ & $x^T x_1 = 0$ contra!