15-859N
10/28/2011

Near $O(m \log n)$ Laplacian solvers

<u>Thm</u> Can find x st if $A\bar{x}=b$ then

$\|x-\bar{x}\|_A < \varepsilon \|\bar{x}\|_A$ in time

$\tilde{O}(m \log n \log(1/\varepsilon))$ expected time.

<u>Thm</u> Can find LSST T st

$\text{stretch}_T(G) = O(m \log n \, ll^3 n)$ in time

$O(m \log n + n \log n \, ll n)$

(AW Thm) Let $Y_1, \ldots, Y_k$ be iid random spsd

$n \times n$ matrices st 1) $E(Y_i) = Z$

2) $Y_i \preceq \mu Z$

$$\Pr\left[(1-\varepsilon)Z \preceq \sum_{i=1}^{k} \frac{Y_i}{k} \preceq (1+\varepsilon)Z\right] \geq 1 - 2n e^{\varepsilon^2 k / 4\mu}$$

Solver has 2 phases

phase $\equiv$ Reduces G to a spine-heavy graph

<u>Def</u> G is spine-heavy if $\exists$ ST T of G st

$$\text{stretch}_T(G \setminus T) = O(m/\log n)$$

Alg (Phase 1) G

1) $T \Leftarrow LSST(G)$
2) $H \Leftarrow G + (t-1)T$ (some constant $t = \kappa \log^2 n$
3) $H' \Leftarrow \text{Sample}(H, tT)$
4) Run $PCG(G, H')$ $O(\log n)$ iterations

using calls to Phase 2(H', b', T).

Sample$(G=(V,E,w), T)$

1) $e \in G \setminus T$ $p'_e = w_e \, ER_T(e)$

2) $t \equiv \sum_{e \in G \setminus T} p'_e \equiv \text{stretch}_T(G \setminus T)$

3) $p_e = p'_e / t$

4) set $k = C_s \, t \log n \log(1/\varepsilon)$

5) Sample k edge $G \setminus T$ with prob p_e

6) return (Sample$/t + T$)

<u>Claim</u> If Phase 2 correctly solves $Hx = b'$ then Phase 1 returns an approx solution to $Gx = b$

Pf $\tau(G/H) = 1$
$\tau(G/H) \le k \log^2 n$ $\Rightarrow \kappa(G, H) \le k \log^2 n$

$\kappa(H, H') \le 3$

$\Rightarrow \kappa(G, H') \le 3k \log^2 n$

Since PCG only requires $O(\kappa(G, H'))$ iterations per bit. $\#iter = O(\log n)$

<u>Claim</u> Let l be a normalized sample returned by Sample(G,T) then

$$\text{stretch}_T(l) = \frac{1}{C_S \log t \log(1/\varepsilon)}$$

<u>Pf</u>

Let $e \in G \setminus T$

$$P_e = P_e'/t$$

if picked weight $\frac{w_e}{P_e'}$

normalized weight $\frac{w_e}{P_e'} \frac{1}{q} = w_l$

$$w_l = \frac{\cancel{w_e}\, \cancel{K}}{\cancel{w_e}\, ER_T(e)} \cdot \frac{1}{C_S \cancel{K} \log K \log(1/\varepsilon)}$$

$$= \frac{1}{C_S\, ER_T(e) \log K \log(1/\varepsilon)}$$

$$\text{stretch}_T(\ell) = w_\ell \, \vec{ER}_T(e) = \frac{1}{C_S \log t \ln(1/\varepsilon)}$$

Phase2(H, T)

1) $H' \leftarrow H + 2T$
2) $H'' = \text{Sample}(H', 3T)$
3) Chebyshev(H, H'')
4) Phase2(H'', T)