

The Basic Iterative Method

Goal: Solve $Au = b$ (*)

Let $u^{(0)}$ our initial guess eg $u^{(0)} = 0$

Richardson's Method

view 1 $u^{(m+1)} = u^{(m)} + (b - Au^{(m)})$

↑
residual error

view 2 $u^{(m+1)} = (I - A)u^{(m)} + b$

$$u^{(m+1)} = G u^{(m)} + b \quad G = (I - A)$$

Good: 1) $O(w(A))$ work per step. (parallel)
2) A fixed point is a solution

Bad: 1) May not converge.

2) If it does it may do so very slowly

The error for Richardson

Suppose $A\bar{u} = b$ & $G = (I - A)$

Def the error $\varepsilon^{(m)} = u^{(m)} - \bar{u}$

Claim $\varepsilon^{(m)} = G^m \varepsilon^{(0)}$

to show $\varepsilon^{(m+1)} = G \varepsilon^{(m)}$

$$\begin{aligned}
 \text{A)} \quad G \varepsilon^{(m)} &= G(u^{(m)} - \bar{u}) = Gu^{(m)} - (I - A)\bar{u} \\
 &= Gu^{(m)} - \bar{u} + A\bar{u} \\
 &= Gu^{(m)} - \bar{u} + b \\
 &= (Gu^{(m)} + b) - \bar{u} \\
 &= u^{(m+1)} - \bar{u} = \varepsilon^{(m+1)}
 \end{aligned}$$

To converge $\lim_{m \rightarrow \infty} G^m \varepsilon^{(0)} = 0$

If it is to converge $\forall \varepsilon^{(0)}$ then $\lim_{m \rightarrow \infty} G^m = 0$

If A is sym then G is sym

$$G = \sum \lambda_i (v_i v_i^T) \quad \lambda_i \in \lambda(G) \quad v_i \text{ eigen vector}$$

IFF $\forall i: |\lambda_i| < 1$ Spectral radius

In general this is ~~not~~ true!

A simple modification to get convergence.

The Extrapolated Method

Goal $Ax=b$ iff $\gamma Ax = \gamma b$

new recurrence $u^{(m+1)} = u^{(m)} + \gamma(b - Ax)$

or $u^{(m+1)} = (I - \gamma A)u^{(m)} + \gamma b$

Let $M_A = \max$ eigenvalue of A .

Let $m_A = \min$ " "

Set $G_\gamma = (I - \gamma A)$ $\gamma = \frac{2}{M_A + m_A}$

$$M_{G_\gamma} = 1 - \left(\frac{2}{M_A + m_A} \right) m_A = \frac{M_A + m_A - 2m_A}{M_A + m_A} = \frac{M_A - m_A}{M_A + m_A}$$

$$m_{G_\gamma} = 1 - \left(\frac{2}{M_A + m_A} \right) M_A = \frac{m_A - M_A}{M_A + m_A} = -M_{G_\gamma}$$

$$\sigma(G_\gamma) = \frac{M_A - m_A}{M_A + m_A}$$

A spd.
Def condition number of $A \equiv \frac{M_A}{m_A} = K(A)$

$$\nabla(G_8) = \frac{K(A)-1}{K(A)+1} \quad K = K(A)$$

$$1 - \frac{2}{K+1} = 1 - \frac{1}{K+1/2} \approx 1 - \frac{1}{K}$$

If we do K iterations, Error $\approx \left(1 - \frac{1}{K}\right)^K \approx \frac{1}{e}$

We need K iterations per bit of accuracy.

EG $A = L(P_n) \quad K(A) \approx n^2$

n^2 iterations per bit.

Polynomial Acceleration

$$Ax=b \quad \text{spd} \quad A\bar{x}=b$$

$$x^{(m+1)} = Gx^{(m)} + b \quad G = (I-A)$$

$$\varepsilon^{(m)} = G^m \varepsilon^{(0)}$$

Idea: Use all previous $x^{(i)}$.

Compute $x^{(0)}, \dots, x^{(n)}$

pick $\alpha_i \in \mathbb{R}$ st. $\sum \alpha_i = 1$

$$u^{(n)} = \sum \alpha_i x^{(i)}$$

Question: How to pick the α_i

$$\begin{aligned} \tilde{\varepsilon}^{(n)} &= u^{(n)} - \bar{x} \\ &= \sum_{i=0}^n \alpha_i x^{(i)} - \bar{x} \quad \sum \alpha_i = 1 \\ &= \sum \alpha_i (x^{(i)} - \bar{x}) \\ &= \sum \alpha_i \varepsilon^{(i)} = \sum \alpha_i G^i \varepsilon^{(0)} \\ &= \left(\sum \alpha_i G^i \right) \tilde{\varepsilon}^{(0)} = Q_n(G) \tilde{\varepsilon}^{(0)} \end{aligned}$$

$$Q_n(\beta) = \alpha_0 + \alpha_1 \beta + \dots + \alpha_{n-1} \beta^{n-1} \quad \sum \alpha_i = 1$$

Goal: For each n pick $Q_n(\beta)$ to

$$\min \|Q_n(G)\| \text{ or } \min \nabla(Q_n(G))$$

Chebyshev Acceleration

The Chebyshev Polys

$$T_0(w) = 1 \quad T_1(w) = w$$

$$T_{n+1}(w) = 2wT_n(w) - T_{n-1}(w)$$

eg $T_2(w) = 2w \cdot w - 1 = 2w^2 - 1$

$$T_3(w) = 2w(2w^2 - 1) - w$$

$$= 4w^3 - 2w - w = 4w^3 - 3w$$

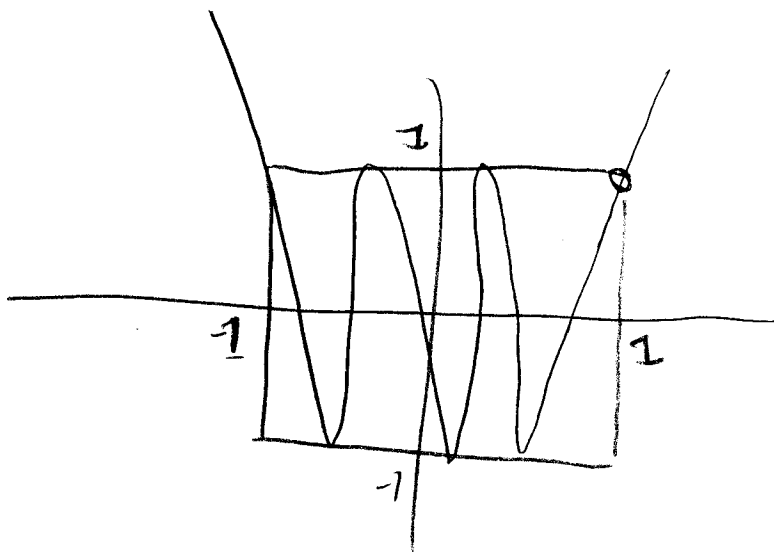
Note $T_n(1) = 1 \Rightarrow \sum \text{coef} = 1$

$$T_n(-1) = (-1)^n$$

$$T_n(w) = \frac{1}{2} \left[(w + \sqrt{w^2 - 1})^n + (w - \sqrt{w^2 - 1})^n \right]$$

HW!

T_n even



Thm $d > 1$ Let $H_n(w) = T_n(w) / T_n(d)$ then

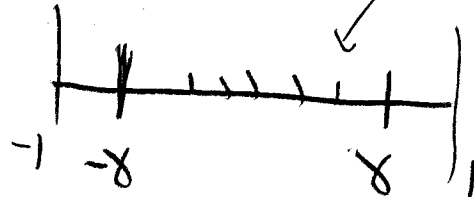
1) $\max_{-1 \leq w \leq 1} |H_n(w)| = 1/T_n(d)$

2) $H_n(w)$ is min such poly

Chebyshev & extrapolated method.

$$K(A) = K \quad \& \quad \gamma = \frac{K-1}{K+1}$$

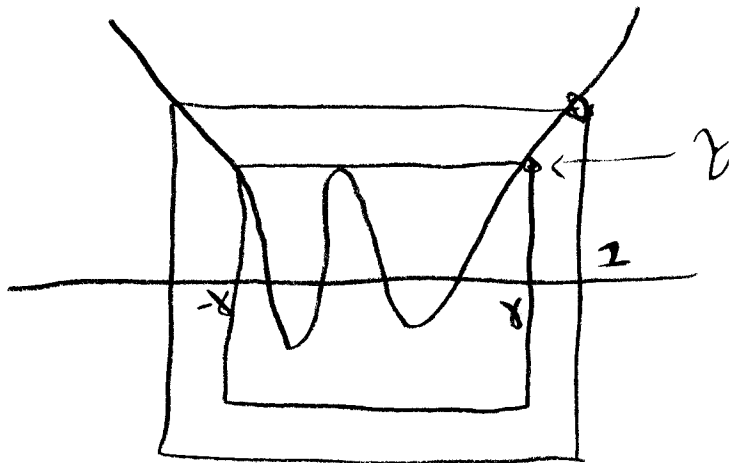
$\lambda(G)$



Eigenvalues of G_0

Pick $\bar{T}_n = T_n(x/\gamma) / T_n(1/\gamma)$

$$\bar{T}_n(1) = 1 \quad \sum \text{coef} = 1$$



Using Thm: $1 = 1/\gamma \Rightarrow x = \gamma$

$$\gamma = 1/T_n(1/\gamma)$$

Goal b.d $T_n(1/k)$

$$1/k = \frac{k+1}{k-1} = 1 + \frac{2}{k-1} = 1 + 2\mu \quad \text{if } \mu = \frac{1}{k-1}$$

$$T_n(x) = \frac{1}{2} \left[(x + \sqrt{x^2 - 1})^n + (x - \sqrt{x^2 - 1})^{-n} \right]$$

$$\geq \frac{1}{2} (x + \sqrt{x^2 - 1})^n \quad x \geq 1$$

$$\begin{aligned} T_n(1+2\mu) &\geq \frac{1}{2} \left(1+2\mu + \sqrt{(1+2\mu)^2 - 1} \right)^n \\ &= \frac{1}{2} \left(1+2\mu + 2\sqrt{\mu(\mu+1)} \right)^n \\ &= \frac{1}{2} \left(\sqrt{\mu} + \sqrt{\mu+1} \right)^{2\mu} \quad (*) \end{aligned}$$

$$\left(\sqrt{\mu} + \sqrt{\mu+1} \right)^2 = \left(\frac{1}{\sqrt{k-1}} + \frac{\sqrt{k}}{\sqrt{k-1}} \right)^2 = \frac{(1+\sqrt{k})^2}{k-1} = \frac{\sqrt{k}+1}{\sqrt{k}-1}$$

$$(*) = \frac{1}{2} \left(\frac{\sqrt{k}+1}{\sqrt{k}-1} \right)^n$$

$$T_n(1/8) \geq \frac{1}{2} \left(\frac{\sqrt{k}+1}{\sqrt{k}-1} \right)^n$$

$$\gamma \leq 2 \left(\frac{\sqrt{k}-1}{\sqrt{k}+1} \right)^n$$

Thm Convergence Rate for Extrapolated Chebyshev
is $\Theta(1/\sqrt{k})$

Back to P_n $K(L(P_n)) = n^2$

$\therefore$ Convergence Rate for Chebyshev is

$$\Theta(1/n)$$

ie n iterations per bit.

This is optimal given that Chebyshev only does
"local" operations!

From Polynomial Recurrence to Iterative Alg

Consider a poly defined by

$$Q_0(x) = 1$$

$$Q_1(x) = \alpha_1 x + \beta_1 \quad \alpha_1 + \beta_1 = 1$$

$$Q_{n+1}(x) = \alpha_n x Q_n(x) + \beta_n Q_{n-1}(x) \quad \alpha_n + \beta_n = 1$$

Iterative Alg

$$u^{(n+1)} = \alpha_n (G u^{(n)} + b) + \beta_n u^{(n-1)}$$

to show $\varepsilon^{(n)} = u^{(n)} - \bar{u}$ then $\varepsilon^{(n)} = Q_n(G) \varepsilon^{(0)}$

$$Q_{n+1}(G) \varepsilon^{(0)} = [\alpha_n G Q_n(G) + \beta_n Q_{n-1}(G)] \varepsilon^{(0)}$$

$$= \alpha_n G \underbrace{Q_n(G) \varepsilon^{(0)}} + \beta_n Q_{n-1}(G) \varepsilon^{(0)}$$

$$= \alpha_n G(\varepsilon^{(n)}) + \beta_n (\varepsilon^{(n-1)})$$

$$= \alpha_n G(u^{(n)} - \bar{u}) + \beta_n (u^{(n-1)} - \bar{u})$$

$$= \alpha_n (Gu^{(n)} - \bar{u} + b) + \beta_n u^{(n-1)} - \beta \bar{u}$$

$$= \alpha_n (Gu^{(n)} + b) + \beta_n u^{(n-1)} - \bar{u}$$

$$= u^{(n+1)} - \bar{u} = \varepsilon^{(n+1)}$$