Robust Regression

Method of least absolute deviation (l_1 -regression)

- Find x* that minimizes $|Ax-b|_1 = \Sigma\ |b_i - <A_{i*}, x>|$
- Cost is less sensitive to outliers than least squares
- Can solve via linear programming

Solving l_1 -regression via Linear Programming

- Minimize $(1,\ldots,1) \cdot (\alpha^+ + \alpha^-)$
- Subject to:

$$A x + \alpha^+ - \alpha^- = b$$
$$\alpha^+, \alpha^- \geq 0$$

- Generic linear programming gives poly(nd) time
- Want much faster time using sketching!

Well-Conditioned Bases

- For an n x d matrix A, can choose an n x d matrix U with orthonormal columns for which A = UW, and $|Ux|_2 = |x|_2$ for all x

- Can we find a U for which A = UW and $|Ux|_1 \approx |x|_1$ for all x?

- Let A = QW where Q has full column rank, and define $|z|_{Q,1} = |Qz|_1$
  - $|z|_{Q,1}$ is a norm

- Let C = $\{z \in R^d : |z|_{Q,1} \leq 1\}$ be the unit ball of $|.|_{Q,1}$

- C is a convex set which is symmetric about the origin
  - Lowner-John Theorem: can find an ellipsoid E such that: $E \subseteq C \subseteq \sqrt{d}E$, where E = $\{z \in R^d : z^T F z \leq 1\}$
  - $(z^T F z)^{.5} \leq |z|_{Q,1} \leq \sqrt{d}(z^T F z)^{.5}$
  - $F = GG^T$ since F defines an ellipsoid

- Define $U = QG^{-1}$

Well-Conditioned Bases

- Recall $U = QG^{-1}$ where
  $\left(z^T F z\right)^{.5} \leq |z|_{Q,1} \leq \sqrt{d}\left(z^T F z\right)^{.5}$ and $F = GG^T$

- $|Ux|_1 = |QG^{-1}x|_1 = |Qz|_1 = |z|_{Q,1}$ where $z = G^{-1}x$

- $z^T F z \quad = \left(x^T \left(G^{-1}\right)^T G^T G \left(G^{-1}\right) x\right) = x^T x = |x|_2^2$

- So $|x|_2 \leq |Ux|_1 \leq \sqrt{d}|x|_2$

- So $\frac{|x|_1}{\sqrt{d}} \leq |x|_2 \leq \left|Ux\right|_1 \leq \sqrt{d}\left|x\right|_2 \leq \sqrt{d}|x|_1$

Net for ℓ_1 – Ball

- Consider the unit ℓ_1-ball B = $\{x \in R^d : |x|_1 = 1\}$
- Subset N is a γ-net if for all $x \in B$, there is a $y \in N$, such that $|x - y|_1 \leq \gamma$
- Greedy construction of N
  - While there is a point $x \in B$ of distance larger than γ from every point in N, include x in N
- The ℓ_1-ball of radius γ/2 around every point in N is contained in the ℓ_1-ball of radius 1+ γ/2 around 0^d
- Further, all such ball are disjoint
- Ratio of volume of d-dimensional similar polytopes of radius 1+ γ/2 to radius $\gamma/2$ is $(1+\gamma/2)^d/(\gamma/2)^d$, so $|N| \leq (1+\gamma/2)^d/(\gamma/2)^d$

Net for $\ell_1 - \mathrm{Subspace}$

- Let A = UW for a well-conditioned basis U
  - $|x|_1 \leq |Ux|_1 \leq d|x|_1$ for all x
- Let N be a (γ/d) $-$net for the unit ℓ_1-ball B
- Let M = {Ux | x in N}, so $|M| \leq (1 + \gamma/(2d))^d/(\gamma/(2d))^d$
- Claim: For every x in B, there is a y in M for which $|Ux - y|_1 \leq \gamma$
- Proof: Let x' in N be such that $|x - x'|_1 \leq \gamma/d$

  Then $|Ux - Ux'|_1 \leq d|x - x'|_1 \leq \gamma$, using the well-conditioned basis property. Set y = Ux'
- $|M| \leq \left(\frac{d}{\gamma}\right)^{O(d)}$

Rough Algorithm Overview

$\min_{x \text{ in } R^d} |Ax-b|_1 = \min_{x \text{ in } R^d} |Ux - b'|_1$

Sample poly(d/ε) rows of $U \circ b'$ proportional to their l_1-norm.

Compute poly(d)-approximation

Compute well-conditioned basis

Find x' such that $|Ax'-b|_1 \leq \text{poly}(d) \min_{x \text{ in } R^d}$ ...
Let b' = b-Ax' be the residual

Find a basis A=UW so that for all x in R^d,
$|x|_1/\text{poly}(d) \leq |Ux|_1 \leq \text{poly}(d) |x|_1$

Takes nnz(A) t...

Now generic linear programming is efficient

Solve l_1-regression on the sample, obtaining vector x, and output x

Will focus on showing how to quickly compute

1. A poly(d)-approximation

2. A well-conditioned basis

Sketching Theorem

Theorem

- There is a probability space over $(d \log d) \times n$ matrices R such that for any $n \times d$ matrix A, with probability at least 99/100 we have for all x:

$$|Ax|_1 \leq |RAx|_1 \leq d \log d \cdot |Ax|_1$$

Embedding

- is linear
- is independent of A
- preserves lengths of an infinite number of vectors

Application of Sketching Theorem

Computing a d(log d)-approximation

- Compute RA and Rb
- Solve x' = $\text{argmin}_x |RAx-Rb|_1$
- Main theorem applied to A∘b implies x' is a d log d – approximation
- RA, Rb have d log d rows, so can solve l_1-regression efficiently

Application of Sketching Theorem

Computing a well-conditioned basis

1. Compute RA
2. Compute W so that RAW is orthonormal (in the l_2-sense)
3. Output U = AW

U = AW is well-conditioned because

$$|AWx|_1 \leq |RAWx|_1 \leq (d \log d)^{1/2} |RAWx|_2 = (d \log d)^{1/2} |x|_2 \leq (d \log d)^{1/2} |x|_1$$

and

$$|AWx|_1 \geq |RAWx|_1/(d \log d) \geq |RAWx|_2/(d \log d) = |x|_2/(d \log d) \geq |x|_1 /(d^{3/2} \log d)$$

Sketching Theorem

Theorem:

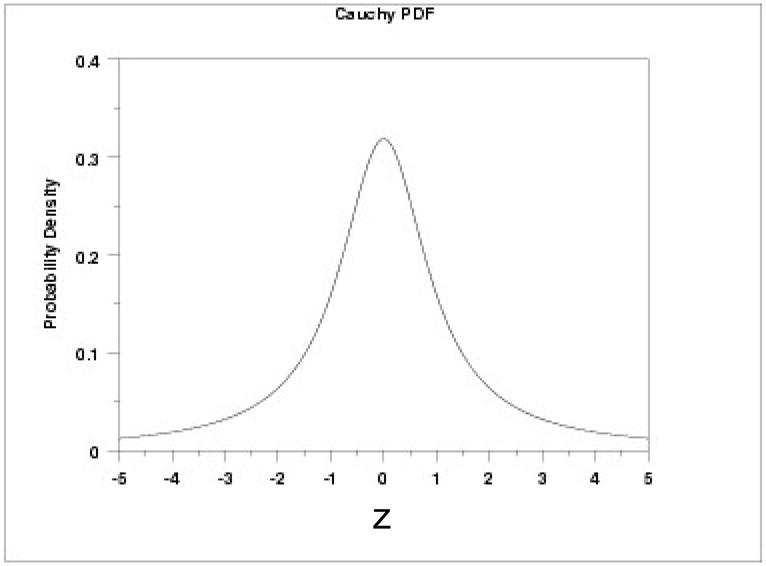
- There is a probability space over (d log d) $\times$ n matrices R such that for any n$\times$d matrix A, with probability at least 99/100 we have for all x:

$$|Ax|_1 \leq |RAx|_1 \leq d \log d \cdot |Ax|_1$$

A dense R that works:

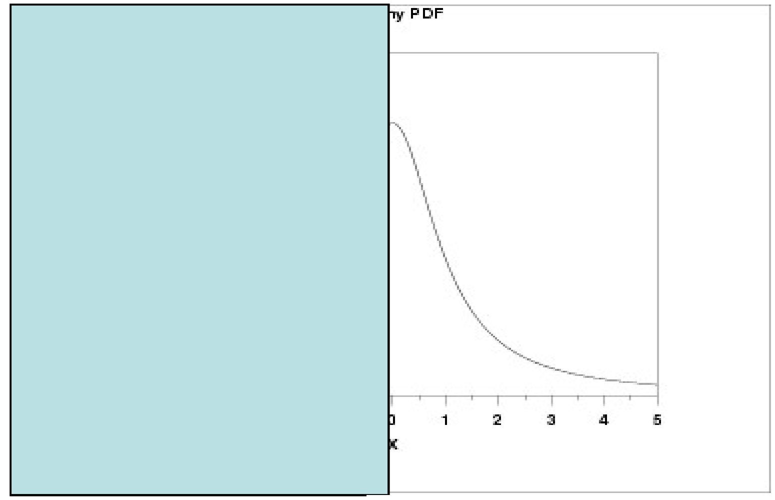
The entries of R are i.i.d. Cauchy random variables, scaled by 1/(d log d)

Cauchy Random Variables



- pdf(z) = $1/(\pi(1+z^2))$ for z in $(-\infty, \infty)$
- Undefined expectation and infinite variance
- 1-stable:
  - If $z_1, z_2, \ldots, z_n$ are i.i.d. Cauchy, then for $a \in R^n$,
    $$a_1 \cdot z_1 + a_2 \cdot z_2 + \ldots + a_n \cdot z_n \sim |a|_1 \cdot z$$, where z is Cauchy
- Can generate as the ratio of two standard normal random variables

Proof of Sketching Theorem



Z

- By 1-stability,
  - For all rows r of R,
    - $\langle r, Ax \rangle = |Ax|_1 \cdot Z / (d \log d)$,
      where Z is a Cauchy
- $RAx = (|Ax|_1 \cdot Z_1, \ldots, |Ax|_1 \cdot Z_{d \log d}) / (d \log d)$,
  where $Z_1, \ldots, Z_{d \log d}$ are i.i.d. Cauchy
- $|RAx|_1 = |Ax|_1 \sum_j |Z_j| / (d \log d)$
  - The $|Z_j|$ are half-Cauchy
- $\sum_j |Z_j| = \Omega(d \log d)$ with probability 1-exp(-d log d) by Chernoff
- But the $|Z_j|$ are heavy-tailed…

Proof of Sketching Theorem

- $\sum_j |Z_j|$ is heavy-tailed, so $|RAx|_1 = |Ax|_1 \sum_j |Z_j| / (d \log d)$ may be large

- Each $|Z_j|$ has c.d.f. asymptotic to $1-\Theta(1/z)$ for z in $[0, \infty)$

- There *exists* a well-conditioned basis of A
  - Suppose w.l.o.g. the basis vectors are $A_{*1}, \ldots, A_{*d}$

- $|RA_{*i}|_1 = |A_{*i}|_1 \cdot \sum_j |Z_{i,j}| / (d \log d)$

- Let $E_{i,j}$ be the event that $|Z_{i,j}| \leq d^3$
  - Define $Z'_{i,j} = |Z_{i,j}|$ if $|Z_{i,j}| \leq d^3$, and $Z'_{i,j} = d^3$ otherwise
  - $E\left[Z_{i,j} \mid E_{i,j}\right] = E\left[Z'_{i,j} \mid E_{i,j}\right] = O(\log d)$

- Let E be the event that for all i,j, $E_{i,j}$ occurs
  - $\Pr[E] \geq 1 - \frac{\log d}{d}$
- What is $E\left[Z'_{i,j} \mid E\right]$?

Proof of Sketching Theorem

- What is $E[Z'_{i,j} \mid E]$?

- $$E[Z'_{i,j} | E_{i,j}] = E[Z'_{i,j} | E_{i,j}, E] \Pr[E \mid E_{i,j}] + E[Z'_{i,j} | E_{i,j}, \neg E] \Pr[\neg E \mid E_{i,j}]$$
$$\geq E[Z'_{i,j} | E_{i,j}, E] \Pr[E \mid E_{i,j}]$$
$$= E[Z'_{i,j} | E] \cdot \left(\frac{\Pr[E_{i,j} | E] \Pr[E]}{\Pr[E_{i,j}]} \right)$$
$$\geq E[Z'_{i,j} | E] \cdot \left(1 - \frac{\log d}{d} \right)$$

- So, $E[Z'_{i,j} | E] = O(\log d)$
- $|RA_{*i}|_1 = |A_{*i}|_1 \cdot \sum_j |Z_{i,j}| / (d \log d)$
- With constant probability, $\sum_i |RA_{*i}|_1 = O(\log d) \sum_i |A_{*i}|_1$

Proof of Sketching Theorem

- With constant probability, $\sum_i |RA_{*i}|_1 = O(\log d) \sum_i |A_{*i}|_1$
- Recall $A_{*1}, \ldots, A_{*d}$ is a well-conditioned basis, and we showed the existence of such a basis earlier
- We will use the Auerbach basis which always exists:
  - For all x, $|x|_\infty \leq |Ax|_1$
  - $\sum_i |A_{*i}|_1 = d$
- $\sum_i |RA_{*i}|_1 = O(d \log d)$
- For all x, $|RAx|_1 \leq \sum_i |RA_{*i} x_i| \leq |x|_\infty \sum_i |RA_{*i}|_1$

  $= |x|_\infty O(d \log d)$

  $= O(d \log d) |Ax|_1$

Where are we?

- Suffices to show for all x with $|x|_1 = 1$, that $|Ax|_1 \le |RAx|_1 \le d \log d \cdot |Ax|_1$
- We know
  - (1) there is a γ-net M, with $|M| \le \left(\frac{d}{\gamma}\right)^{O(d)}$, of the set $\{Ax \text{ such that } |x|_1 = 1\}$
  - (2) for any fixed x, $|RAx|_1 \ge |Ax|_1$ with probability $1 - \exp(-d \log d)$
  - (3) for all x, $|RAx|_1 = O(d \log d)|Ax|_1$

- Set $\gamma = 1/(d^3 \log d)$ so $|M| \le d^{O(d)}$
  - By a union bound, for all y in M, $|Ry|_1 \ge |y|_1$

- Let x with $|x|_1 = 1$ be arbitrary. Let y in M satisfy $|Ax - y|_1 \le \gamma = 1/(d^3 \log d)$

- $|RAx|_1 \ge |Ry|_1 - |R(Ax - y)|_1$

  $\ge |y|_1 - O(d \log d)|Ax - y|_1$

  $\ge |y|_1 - O(d \log d)\gamma$

  $\ge |y|_1 - O\left(\frac{1}{d^2}\right)$

  $\ge |y|_1/2$ (why?)

Outline

- Quick recap of ℓ_1-regression, and how to speed it up
- Introduction to the Streaming Model and Estimating Norms

L_1 Regression Algorithm Recap

Compute poly(d)-approximation

Compute well-conditioned basis

Sample rows from the well-conditioned basis and the residual of the poly(d)-approximation

Solve l_1-regression on the sample, obtaining vector x, and output x

We saw how to solve the above problems by sketching by a matrix of i.i.d. Cauchy random variables

Sketching to solve l_1-regression [CW, MM]

- Most expensive operation is computing R*A where R is the matrix of i.i.d. Cauchy random variables

- All other operations are in the “smaller space”

- Can speed this up by choosing R as follows:

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} C_1 & & & & \\ & C_2 & & & \\ & & C_3 & & \\ & & & \cdots & \\ & & & & C_n \end{bmatrix}$$

- For all x, $\left(\frac{1}{d^2 \log^2 d}\right) |Ax|_1 \le |RAx|_1 \le O(d \log d)\, |Ax|_1$
- Overall time for ℓ_1-regression is nnz(A) + poly(d/ϵ)

Further sketching improvements [WZ]

- Can show you need a fewer number of sampled rows in later steps if instead choose R as follows

- Instead of diagonal of Cauchy random variables, choose diagonal of reciprocals of exponential random variables

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1/E_1 & & & & \\ & 1/E_2 & & & \\ & & 1/E_3 & & \\ & & & \cdots & \\ & & & & 1/E_n \end{bmatrix}$$

- For all x, $\left(\frac{1}{d^{.5}\mathrm{poly}(\log(nd))}\right)|Ax|_1 \le |RAx|_1 \le O(d \log d)\, |Ax|_1$

Fun Fact about Cauchy Random Variables

- Suppose you have i.i.d. copies $R_1, \dots, R_n$ of a random variable with mean 0 and variance σ^2
- What is the distribution of $\frac{\sum_i R_i}{n}$?
- By Central Limit Theorem, this approaches a normal random variable N(0, $\sigma^2/n)$
- Intuitively, the variance is decreasing and the average is approaching its expectation
- Now suppose you have i.i.d. copies $R_1, \dots, R_n$ of a standard Cauchy random variable
- What is the distribution of $\frac{\sum_i R_i}{n}$?
- It's still a standard Cauchy random variable!

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Turnstile Streaming Model

- Underlying n-dimensional vector x initialized to 0^n
- Long stream of updates $x_i \leftarrow x_i + \Delta_j$ for Δ_j in {-M, -M+1, …, M-1, M}
  - $M \leq \mathrm{poly}(n)$
- Throughout the stream, x is promised to be in $\{-M, -M+1, \ldots, M-1, M\}^n$
- Output an approximation to f(x) with high probability over our coin tosses
- Goal: use as little space (in bits) as possible
  - Massive data: stock transactions, weather data, genomes

Testing if x = 0^n

- How can we test, with probability at least 9/10, over our random coin tosses, if the underlying vector $x = 0^n$?

- Can we use O(log n) bits of space?

- We saw that for any fixed vector x, if S is a CountSketch matrix with $O(\frac{1}{\epsilon^2})$ rows, then $|Sx|_2^2 = (1 \pm \epsilon)|x|_2^2$ with probability at least 9/10

- If we set $\epsilon = \frac{1}{2}$, we use O(log n) bits of space to store the O(1) entries of Sx

- We can store the hash function and sign function defining S using O(log n) bits

Testing if x = 0^n

- Is there a deterministic, i.e., zero-error, streaming algorithm to test if the underlying vector $x = 0^n$ with o(n log n) bits of space?
- Theorem: any deterministic algorithm requires $\Omega(n \log n)$ bits of space
- Suppose the first half of the stream corresponds to updates to a vector a in $\{0, 1, 2, \ldots, \mathrm{poly}(n)\}^n$
- Let S(a) be the state of the algorithm after reading the first half of the stream
  - If |S(a)| = o(n log n), there exist a$\neq$ a′ for which S(a) = S(a')
- Suppose the second half of the stream corresponds to updates to a vector b in $\{0, -1, -2, \ldots, -\mathrm{poly}(n)\}^n$
- The algorithm must output the same answer on a+b and a'+b, so it errs in one case

Example: Recovering a k-Sparse Vector

- Suppose we are promised that x has at most k non-zero entries at the end of the stream

- k is often small – maybe we see all coordinates of a vector a followed by all coordinates of a *similar* vector b, and a-b only has k non-zero entries

- Can we recover the indices and values of the k non-zero entries with high probability?

- Can we use k poly(log n) bits of space?

- Can we do it deterministically?

Example: Recovering a k-Sparse Vector

- Suppose A is an s x n matrix such that any 2k columns are linearly independent
- Maintain $A \cdot x$ in the stream
- Claim: from $A \cdot x$ you can recover the subset S of k non-zero entries and their values
- Proof: suppose there were vectors x and y each with at most k non-zero entries and $A \cdot x = A \cdot y$
- Then A(x-y) = 0. But x-y has at most 2k non-zero entries, and any 2k columns of A are linearly independent. So x-y = 0, i.e., x = y.
- Algorithm is deterministic given A. But do such matrices A exist with a small number s of rows?

Example: Recovering a k-Sparse Vector

- Vandermonde matrix A with s = 2k rows and n columns. $A_{i,j} = j^{i-1}$

$$\begin{bmatrix} 1 & 1 & 1 & \dots \\ 1 & 2 & 3 & \dots \\ 1 & 4 & 9 & \dots \\ 1 & 8 & 27 & \dots \end{bmatrix}$$

- Determinant of 2k x 2k submatrix of A with set of columns equal to $\{i_1, \ldots, i_{2k}\}$ is: $\prod_j i_j \prod_{j<j'} (i_j - i_{j'}) \neq 0$, so any 2k columns of A are linearly independent

- But entries of A are exponentially increasing – how to store A and $A \cdot x$?

- Just store $A \cdot x$ mod p for a large enough prime p = poly(n)

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Example Problem: Norms

- Suppose you want $|x|_p^p = \Sigma_{i=1}^n |x_i|^p$
- Want Z for which $(1-\varepsilon) |x|_p^p \leq Z \leq (1+\varepsilon) |x|_p^p$ with probability > 9/10
- p = 1 corresponds to total variation distance between distributions
- p = 2 useful for geometric and linear algebraic problems
- $p = \infty$ is the value of the maximum entry, useful for anomaly detection, etc.