Nonnegative Matrix Factorization

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\begin{aligned}
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&= \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 2 \\ 3 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}
\end{aligned}
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&= \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 2 \\ 3 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \\
&= UV
\end{aligned}
$$

Main Question, Nonnegative Matrix Factorization

Question

Given matrix $A \in \mathbb{R}^{n \times n}$ and $k \geqslant 1$, is there an algorithm that can determine if there exist two matrices $U, V^\top \in \mathbb{R}^{n \times k}$,

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Or, are there any hardness results?

- Equivalent to computing the nonnegative rank of A, $\mathrm{rank}_+(A)$
- Fundamental question in machine learning
- Applications
  - Text mining, Spectral data analysis, Scalable Internet distance prediction, Non-stationary speech denoising, Bioinformatics, Nuclear imaging, etc.

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$$= \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\mathrm{rank}(A) = 3,$$

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$$\mathrm{rank}(A) = 3, \quad \text{but } \mathrm{rank}_+(A) = 4$$

Polynomial System Verifier

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H : the bitsizes of the coefficients of the polynomials

In $(md)^{O(v)} \operatorname{poly}(H)$ time, can
decide if there exists a solution to polynomial system P

Main Idea

1. Write $\min_{U,V^\top \in \mathbb{R}^{n\times k}, U,V \geqslant 0} \|UV - A\|_F^2$ as a polynomial system that has $\mathrm{poly}(k)$ variables and $\mathrm{poly}(n)$ constraints and degree

2. Use polynomial system verifier to solve it

Algorithm

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in $n^{2^{O(k)}}$ time Arora-Ge-Kannan-Moitra'12

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in $2^{O(k^3)} n^{O(k^2)}$ time Moitra'13

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then k-SUM cannot be solved in $n^{o(k)}$ time

Hardness - Under Exponential Time Hypothesis

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[Impagliazzo-Paturi-Zane'98]

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Assume : Exponential Time Hypothesis
[Impagliazzo-Paturi-Zane'98]

Requires : $n^{\Omega(k)}$ time

Open Problems

- The upper bound is $n^{O(k^2)}$ while the lower bound is $n^{\Omega(k)}$ - what is the right answer?

Weighted Low Rank Approximation

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Output : rank-k $\widehat{A} \in \mathbb{R}^{n \times n}$ s.t.

$$\left\| W \circ (\widehat{A} - A) \right\|_F^2$$

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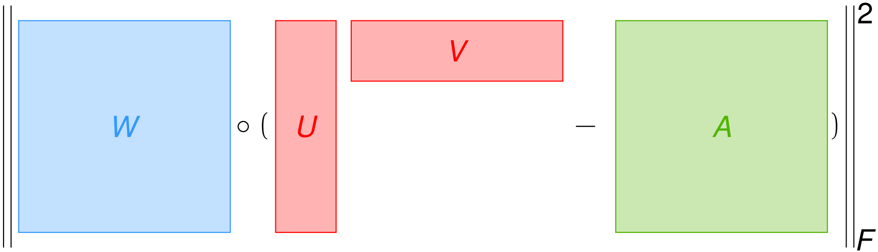

Motivation

Motivation


DIVIDED WE FALL
CIVIL WAR
MAY 6
JIM CARREY
MASK
FROM ZERO TO HERO
HANGOVER
LINCOLN
ZOOTOPIA
LORD OF THE RINGS

Motivation


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Motivation

Action Comedy Historical Cartoon Magical

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9 8

Motivation

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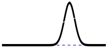

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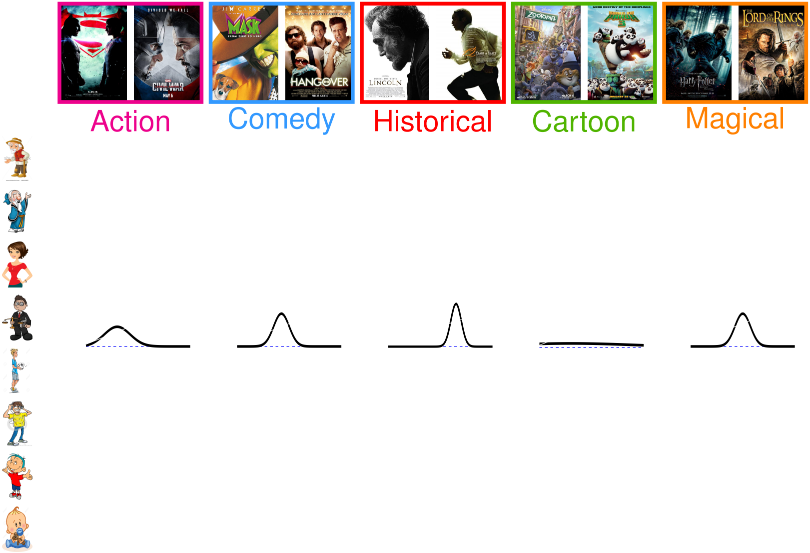

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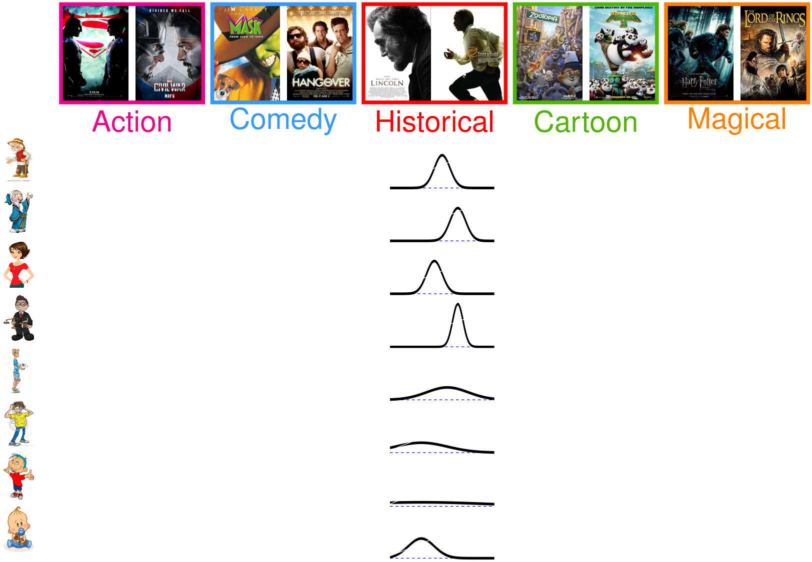

Motivation

Action Comedy Historical Cartoon Magical

$\frac{1}{5}$	$\frac{1}{5}$
$\frac{1}{1}$	$\frac{1}{1}$
$\frac{1}{1}$	$\frac{1}{1}$
$\frac{1}{5}$	$\frac{1}{5}$
$\frac{1}{1}$	$\frac{1}{1}$
$\frac{1}{4}$	$\frac{1}{4}$
$\frac{1}{6}$	$\frac{1}{6}$
$\frac{1}{1}$	$\frac{1}{1}$

Motivation

Action		Comedy		Historical		Cartoon		Magical	
$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{1}$	$\frac{1}{1}$						
$\frac{1}{1}$	$\frac{1}{1}$	$\frac{1}{1}$	$\frac{1}{1}$						
$\frac{1}{1}$	$\frac{1}{1}$	$\frac{1}{1}$	$\frac{1}{1}$						
$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{6}$						
$\frac{1}{1}$	$\frac{1}{1}$	$\frac{1}{3}$	$\frac{1}{3}$						
$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{2}$						
$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{2}$						
$\frac{1}{1}$	$\frac{1}{1}$	$\frac{1}{4}$	$\frac{1}{4}$						

Motivation

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	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{1}$	$\frac{1}{1}$	$\frac{1}{3}$	$\frac{1}{3}$		
	$\frac{1}{1}$	$\frac{1}{1}$	$\frac{1}{1}$	$\frac{1}{1}$	$\frac{1}{6}$	$\frac{1}{6}$		
	$\frac{1}{1}$	$\frac{1}{1}$	$\frac{1}{1}$	$\frac{1}{1}$	$\frac{1}{4}$	$\frac{1}{4}$		
	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{2}$		
	$\frac{1}{1}$	$\frac{1}{1}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{7}$	$\frac{1}{7}$		
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	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$		
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$\frac{1}{1}$	$\frac{1}{1}$	$\frac{1}{1}$	$\frac{1}{1}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{1}$	$\frac{1}{1}$	
$\frac{1}{1}$	$\frac{1}{1}$	$\frac{1}{1}$	$\frac{1}{1}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{2}$	
$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{1}$	$\frac{1}{1}$	
$\frac{1}{1}$	$\frac{1}{1}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{5}$	$\frac{1}{5}$	
$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{1}$	$\frac{1}{1}$	
$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{3}$	
$\frac{1}{1}$	$\frac{1}{1}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{1}$	$\frac{1}{1}$	$\frac{1}{4}$	$\frac{1}{4}$	

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$\frac{1}{1}$	$\frac{1}{1}$	$\frac{1}{1}$	$\frac{1}{1}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{1}$	$\frac{1}{1}$	$\frac{1}{2}$	$\frac{1}{2}$
$\frac{1}{1}$	$\frac{1}{1}$	$\frac{1}{1}$	$\frac{1}{1}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{6}$
$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{1}$	$\frac{1}{1}$	$\frac{1}{4}$	$\frac{1}{4}$
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Motivation

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8 8

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Motivation

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8	8
8	8
4	4
4	4

Motivation

Action Comedy Historical Cartoon Magical

8	8
8	8
4	4
4	4
1	1
1	1

Motivation

Action		Comedy		Historical		Cartoon		Magical	
				8	8				
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				0	0				
				0	0				

Motivation

Action		Comedy		Historical		Cartoon		Magical	
				8	8				
				8	8				
				4	4				
				4	4				
				1	1				
				1	1				
				0	0	8	8		
				0	0	8	8		

Motivation

Action		Comedy		Historical		Cartoon		Magical	
				8	8				
				8	8				
				4	4				
				4	4				
				1	1	4	4		
				1	1	4	4		
				0	0	8	8		
				0	0	8	8		

Motivation

Action Comedy Historical Cartoon Magical

				8	8				
				8	8				
				4	4	2	2		
				4	4	2	2		
				1	1	4	4		
				1	1	4	4		
				0	0	8	8		
				0	0	8	8		

Motivation

Action

Comedy

Historical

Cartoon

Magical

8	8	1	1
8	8	1	1
4	4	2	2
4	4	2	2
1	1	4	4
1	1	4	4
0	0	8	8
0	0	8	8

Matrix Completion

Matrix Completion

Given : $A \in \{\mathbb{R}, ?\}^{n \times n}$, $\Omega \subset [n] \times [n]$ and $k \in \mathbb{R}$

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Given : $A \in \{\mathbb{R}, ?\}^{n \times n}$, $\Omega \subset [n] \times [n]$ and $k \in \mathbb{R}$

$$A = \begin{bmatrix} 1 & ? & ? & 0 \\ ? & 1 & 0 & ? \\ ? & 0 & 1 & ? \\ 0 & ? & ? & 1 \end{bmatrix}, k = 2$$

Matrix Completion

Given : $A \in \{\mathbb{R}, ?\}^{n\times n}$, $\Omega \subset [n] \times [n]$ and $k \in \mathbb{R}$

$$A = \begin{bmatrix} 1 & ? & ? & 0 \\ ? & 1 & 0 & ? \\ ? & 0 & 1 & ? \\ 0 & ? & ? & 1 \end{bmatrix}, k = 2$$

Output : $A_{\bar{\Omega}}$ s.t. $\mathrm{rank}(A) = k$

Matrix Completion

Given : $A \in \{\mathbb{R}, ?\}^{n \times n}$, $\Omega \subset [n] \times [n]$ and $k \in \mathbb{R}$

$$A = \begin{bmatrix} 1 & ? & ? & 0 \\ ? & 1 & 0 & ? \\ ? & 0 & 1 & ? \\ 0 & ? & ? & 1 \end{bmatrix}, k = 2$$

Output : $A_{\bar{\Omega}}$ s.t. $\mathrm{rank}(A) = k$

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \; A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

Matrix Completion

Given : $A \in \{\mathbb{R}, ?\}^{n \times n}$, $\Omega \subset [n] \times [n]$ and $k \in \mathbb{R}$

$$A = \begin{bmatrix} 1 & ? & ? & 0 \\ ? & 1 & 0 & ? \\ ? & 0 & 1 & ? \\ 0 & ? & ? & 1 \end{bmatrix}, k = 2$$

Output : $A_{\bar{\Omega}}$ s.t. $\mathrm{rank}(A) = k$

Algorithms : Candes-Recht'09, Candes-Recht'10, Keshavan'12
Hardt-Wootters'14, Jain-Netrapalli-Sanghavi'13
Hardt'15, Sun-Luo'15

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Algorithms : Candes-Recht'09, Candes-Recht'10, Keshavan'12
Hardt-Wootters'14, Jain-Netrapalli-Sanghavi'13
Hardt'15, Sun-Luo'15

Hardness : Peeters'96, Gillis-Glineur'11
Hardt-Meka-Raghavendra-Weitz'14

More Real-life Datasets

More Real-life Datasets

RateBeer

Tasters

Beers


Heaven Or Helles
Boston Beer Company (Samuel Adams)
Munich Helles Lager / 5.00% ABV

3.71/5 rDev +0.5% | Avg: 3.69
look: 3.25 | smell: 3.5 | taste: 4 | feel: 3.75 | overall: 3.5

Pours a slightly hazy golden straw color with a pillowy 1-1 1/2 inch head that slowly dissipates to slightly less than 1 finger of head with good lacing. The nose had aromas of bread malt and a spiciness I belie cv is pepper. The mouthfeel was lightly carbonated with a richness/sweetness. A slight bitterness on the back end rounded out the flavor palate and struck a nice balance. I found it very satisfying and I'd go as far as saying it would make a nice session beer. The only thing I don't like is that it is a limited release.

★ 533 characters

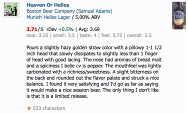
Ratings

More Real-life Datasets

RateBeer

Tasters

Beers

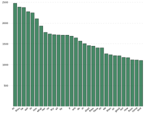


Ratings

Documents

Words

Books



Frequency

More Real-life Datasets

RateBeer

Tasters

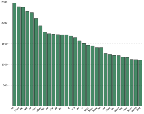
Beers

Ratings

Documents

Words

Books



Frequency

Amazon

Users

Products

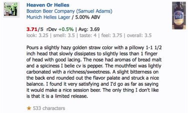
Ratings

More Real-life Datasets

RateBeer

Tasters

Beers



Ratings

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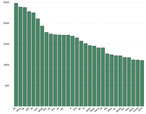


Frequency

Amazon

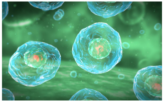
Users

Products

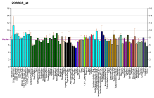
Ratings

Biology

Genes



Cells



Levels

Several Results in [Razenshteyn, Song, W]

- Algorithm for Weighted low rank approximation(WLRA) problem

Several Results in [Razenshteyn, Song, W]

- Algorithm for Weighted low rank approximation(WLRA) problem
  - W has r distinct rows and columns

$$W = \begin{bmatrix} 1 & 1 & 1 & 3 & 3 & 4 \\ 1 & 1 & 1 & 3 & 3 & 4 \\ 2 & 2 & 2 & 5 & 5 & 0 \\ 2 & 2 & 2 & 5 & 5 & 0 \\ 2 & 2 & 2 & 5 & 5 & 0 \\ 7 & 7 & 7 & 4 & 4 & 6 \end{bmatrix}, r = 3$$

Several Results in [Razenshteyn, Song, W]

- Algorithm for Weighted low rank approximation(WLRA) problem
  - W has r distinct rows and columns
  - W has r distinct columns

$$W = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 1 & 1 & 3 \\ 3 & 3 & 3 & 1 & 1 & 5 \\ 4 & 4 & 4 & 1 & 1 & 1 \\ 5 & 5 & 5 & 1 & 1 & 3 \\ 6 & 6 & 6 & 1 & 1 & 5 \end{bmatrix}, r = 3$$

Several Results in [Razenshteyn, Song, W]

- Algorithm for Weighted low rank approximation(WLRA) problem
  - W has r distinct rows and columns
  - W has r distinct columns
  - W has rank at most r

$$W = \begin{bmatrix} 1 & 1 & 1 & 2 & 2 & 2 \\ 0 & 1 & 2 & 1 & 2 & 3 \\ 1 & 1 & 3 & 2 & 4 & 4 \\ 0 & 0 & 4 & 0 & 4 & 4 \\ 1 & 0 & 5 & 1 & 6 & 5 \\ 0 & 0 & 6 & 0 & 6 & 6 \end{bmatrix}, r = 3$$

Several Results in [Razenshteyn, Song, W]

- Algorithm for Weighted low rank approximation(WLRA) problem
  - W has r distinct rows and columns
  - W has r distinct columns
  - W has rank at most r
- Hardness for Weighted low rank approximation(WLRA) problem

r Distinct Rows and Columns

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Given : $A \in \mathbb{R}^{n\times n}$, $W \in \mathbb{R}^{n\times n}$, $r \in \mathbb{N}$, $k \in \mathbb{N}$, $\epsilon > 0$

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Given : $A \in \mathbb{R}^{n\times n}$, $W \in \mathbb{R}^{n\times n}$, $r \in \mathbb{N}$, $k \in \mathbb{N}$, $\epsilon > 0$

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Output :

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Output : rank-k $\widehat{A} \in \mathbb{R}^{n \times n}$ s.t.

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Output : rank-k $\widehat{A} \in \mathbb{R}^{n\times n}$ s.t.

$$\|W \circ (A - \widehat{A})\|_F^2 \leqslant (1+\epsilon) \min_{\text{rank}-k\ A'} \|W \circ (A - A')\|_F^2$$

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Given : $A \in \mathbb{R}^{n \times n}$, $W \in \mathbb{R}^{n \times n}$, $r \in \mathbb{N}$, $k \in \mathbb{N}$, $\epsilon > 0$

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Output : rank-k $\widehat{A} \in \mathbb{R}^{n\times n}$ s.t.
$\| W \circ (A - \widehat{A}) \|_F^2 \leqslant (1+\epsilon)\mathrm{OPT}$
with prob. $9/10$

r Distinct Rows and Columns

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$\| W \circ (A - \widehat{A}) \|_F^2 \leqslant (1 + \epsilon)\mathrm{OPT}$
with prob. $9/10$
in $O((\mathrm{nnz}(A) + \mathrm{nnz}(W)) \cdot n^{\gamma}) + n \cdot 2^{\widetilde{O}(k^2 r/\epsilon)}$ time

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Given : $A \in \mathbb{R}^{n \times n}$, $W \in \mathbb{R}^{n \times n}$, $r \in \mathbb{N}$, $k \in \mathbb{N}$, $\epsilon > 0$

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for an arbitrarily small constant $\gamma > 0$

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Given : $A \in \mathbb{R}^{n\times n}$, $W \in \mathbb{R}^{n\times n}$, $r \in \mathbb{N}$, $k \in \mathbb{N}$, $\epsilon > 0$

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for an arbitrarily small constant $\gamma > 0$
fixed parameter tractable

r Distinct Columns

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Given : $\quad A \in \mathbb{R}^{n \times n}$, $W \in \mathbb{R}^{n \times n}$, $r \in \mathbb{N}$, $k \in \mathbb{N}$, $\epsilon > 0$

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Output :

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$$W = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 1 & 1 & 3 \\ 3 & 3 & 3 & 1 & 1 & 5 \\ 4 & 4 & 4 & 1 & 1 & 1 \\ 5 & 5 & 5 & 1 & 1 & 3 \\ 6 & 6 & 6 & 1 & 1 & 5 \end{bmatrix}, r = 3$$

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Output : rank-k $\widehat{A} \in \mathbb{R}^{n\times n}$ s.t.
$\|W \circ (A - \widehat{A})\|_F^2 \leqslant (1+\epsilon)\mathrm{OPT}$
with prob. $9/10$

r Distinct Columns

Given : $A \in \mathbb{R}^{n \times n}$, $W \in \mathbb{R}^{n \times n}$, $r \in \mathbb{N}$, $k \in \mathbb{N}$, $\epsilon > 0$

$$W = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 1 & 1 & 3 \\ 3 & 3 & 3 & 1 & 1 & 5 \\ 4 & 4 & 4 & 1 & 1 & 1 \\ 5 & 5 & 5 & 1 & 1 & 3 \\ 6 & 6 & 6 & 1 & 1 & 5 \end{bmatrix}, r = 3$$

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with prob. $9/10$
in $O((\mathrm{nnz}(A) + \mathrm{nnz}(W)) \cdot n^{\gamma}) + n \cdot 2^{\widetilde{O}(k^2 r^2 / \epsilon)}$ time

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Given : $A \in \mathbb{R}^{n\times n}$, $W \in \mathbb{R}^{n\times n}$, $r \in \mathbb{N}$, $k \in \mathbb{N}$, $\epsilon > 0$

$$W = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 1 & 1 & 3 \\ 3 & 3 & 3 & 1 & 1 & 5 \\ 4 & 4 & 4 & 1 & 1 & 1 \\ 5 & 5 & 5 & 1 & 1 & 3 \\ 6 & 6 & 6 & 1 & 1 & 5 \end{bmatrix}, r = 3$$

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for an arbitrarily small constant $\gamma > 0$
fixed parameter tractable

Rank r

Rank r

Given : $A \in \mathbb{R}^{n \times n}$, $W \in \mathbb{R}^{n \times n}$, $r \in \mathbb{N}$, $k \in \mathbb{N}$, $\epsilon > 0$

Rank r

Given : $A \in \mathbb{R}^{n\times n}$, $W \in \mathbb{R}^{n\times n}$, $r \in \mathbb{N}$, $k \in \mathbb{N}$, $\epsilon > 0$

$$W = \begin{bmatrix} 1 & 1 & 1 & 2 & 2 & 2 \\ 0 & 1 & 2 & 1 & 2 & 3 \\ 1 & 1 & 3 & 2 & 4 & 4 \\ 0 & 0 & 4 & 0 & 4 & 4 \\ 1 & 0 & 5 & 1 & 6 & 5 \\ 0 & 0 & 6 & 0 & 6 & 6 \end{bmatrix}, r = 3$$

Rank r

Given : $A \in \mathbb{R}^{n\times n}$, $W \in \mathbb{R}^{n\times n}$, $r \in \mathbb{N}$, $k \in \mathbb{N}$, $\epsilon > 0$

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Output :

Rank r

Given : $A \in \mathbb{R}^{n\times n}$, $W \in \mathbb{R}^{n\times n}$, $r \in \mathbb{N}$, $k \in \mathbb{N}$, $\epsilon > 0$

$$W = \begin{bmatrix} 1 & 1 & 1 & 2 & 2 & 2 \\ 0 & 1 & 2 & 1 & 2 & 3 \\ 1 & 1 & 3 & 2 & 4 & 4 \\ 0 & 0 & 4 & 0 & 4 & 4 \\ 1 & 0 & 5 & 1 & 6 & 5 \\ 0 & 0 & 6 & 0 & 6 & 6 \end{bmatrix}, r = 3$$

Output : rank-k $\widehat{A} \in \mathbb{R}^{n\times n}$ s.t.

Rank r

Given : $A \in \mathbb{R}^{n \times n}$, $W \in \mathbb{R}^{n \times n}$, $r \in \mathbb{N}$, $k \in \mathbb{N}$, $\epsilon > 0$

$$W = \begin{bmatrix} 1 & 1 & 1 & 2 & 2 & 2 \\ 0 & 1 & 2 & 1 & 2 & 3 \\ 1 & 1 & 3 & 2 & 4 & 4 \\ 0 & 0 & 4 & 0 & 4 & 4 \\ 1 & 0 & 5 & 1 & 6 & 5 \\ 0 & 0 & 6 & 0 & 6 & 6 \end{bmatrix}, r = 3$$

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previously only r=1 was known to be in polynomial time

Random-4SAT Hypothesis

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[Feige'02, Goerdt-Lanka'04]

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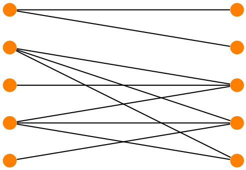

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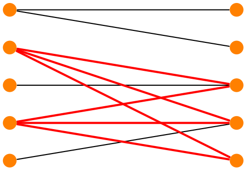

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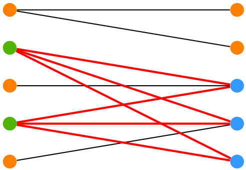

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It takes $(md)^{O(v)} \operatorname{poly}(H)$ time to
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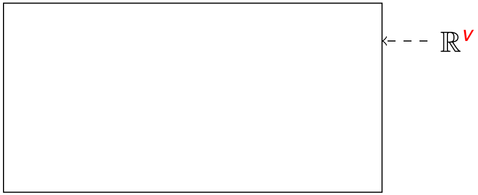

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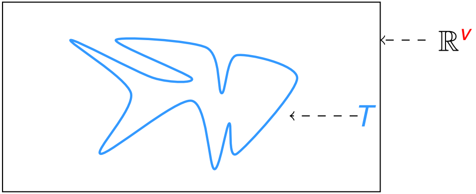

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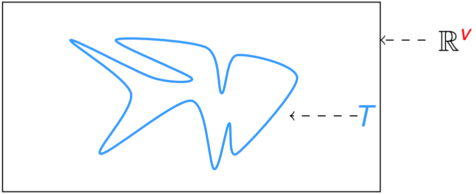
minimum value that nonnegative g takes over $T \cap$ *Ball* is



Lower Bound on the Cost
[Jeronimo-Perrucci-Tsigaridas'13]

Given: a real polynomial system $P(x)$

v : #variables, $x = (x_1, x_2, \cdots, x_v)$
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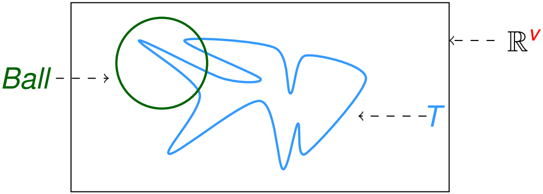

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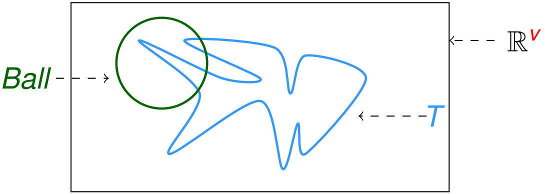

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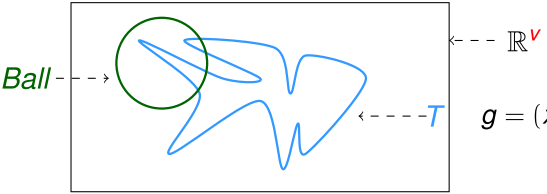

$g = (x_1 x_2 - 1)^2 + x_2^2$

Multiple Regression Sketch

Multiple Regression Sketch

Given :

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Given : $A^{(1)}, A^{(2)}, \cdots, A^{(m)} \in \mathbb{R}^{n \times k}$

Multiple Regression Sketch

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with prob. $9/10$

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Warmup, inefficient WLRA Algorithm

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Given : $A \in \mathbb{R}^{n\times n}$, $W \in \mathbb{R}^{n\times n}$, $r \in \mathbb{N}$, $k \in \mathbb{N}$, $\epsilon > 0$

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polynomial verifier runs in $(\#\,\text{constraints} \cdot \text{degree})^{O(\#\,\text{variables})}$

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lower bound on cost $(\#\,\text{constraints})^{-\text{degree}^{O(\#\,\text{variables})}}$

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write a polynomial with few #variables, i.e. $\mathrm{poly}(kr/\epsilon)$

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lower bound on cost $(\#\,\mathrm{constraints})^{-\mathrm{degree}^{O(\#\,\mathrm{variables})}}$

write a polynomial with few #variables, i.e. $\mathrm{poly}(kr/\epsilon)$

without blowing up degree and #constraints too much

Main Idea

To reduce the number of variables to $\mathrm{poly}(kr/\epsilon)$:

1. Multiple regression sketch with $O(k/\epsilon)$ rows

2. Weight matrix W has rank at most r

Guess a Sketch

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$W_{ij} \in \{0, 1, 2, \cdots, \Delta\}$

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Given : $A \in \mathbb{R}^{n\times n}$, $W \in \mathbb{R}^{n\times n}$, $r \in \mathbb{N}$, $k \in \mathbb{N}$, $\epsilon > 0$

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Algorithm : W_j be jth column of W

D_{W_j} be diagonal matrix with vector W_j

$$W = \begin{bmatrix} 1 & 1 & 1 & 2 & 2 & 2 \\ 0 & 1 & 2 & 1 & 2 & 3 \\ 1 & 1 & 3 & 2 & 4 & 4 \\ 0 & 0 & 4 & 0 & 4 & 4 \\ 1 & 0 & 5 & 1 & 6 & 5 \\ 0 & 0 & 6 & 0 & 6 & 6 \end{bmatrix}$$

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$$W = \begin{bmatrix} \boxed{\begin{matrix}1\\0\\1\\0\\1\\0\end{matrix}} & \begin{matrix}1\\1\\1\\0\\0\\0\end{matrix} & \begin{matrix}1\\2\\3\\4\\5\\6\end{matrix} & \begin{matrix}2\\1\\2\\0\\1\\0\end{matrix} & \begin{matrix}2\\2\\4\\4\\6\\6\end{matrix} & \begin{matrix}2\\3\\4\\4\\5\\6\end{matrix} \end{bmatrix} \quad D_{W_1} = \begin{bmatrix} 1 & & & & & \\ & 0 & & & & \\ & & 1 & & & \\ & & & 0 & & \\ & & & & 1 & \\ & & & & & 0 \end{bmatrix}$$

Guess a Sketch

Given : $A \in \mathbb{R}^{n \times n}$, $W \in \mathbb{R}^{n \times n}$, $r \in \mathbb{N}$, $k \in \mathbb{N}$, $\epsilon > 0$

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Guess a Sketch

Given : $A \in \mathbb{R}^{n\times n}$, $W \in \mathbb{R}^{n\times n}$, $r \in \mathbb{N}$, $k \in \mathbb{N}$, $\epsilon > 0$

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Guess a Sketch

Given : $A \in \mathbb{R}^{n \times n}$, $W \in \mathbb{R}^{n \times n}$, $r \in \mathbb{N}$, $k \in \mathbb{N}$, $\epsilon > 0$

Output : $U, V^\top \in \mathbb{R}^{n \times k}$ s.t. $\| W \circ (UV - A) \|_F^2 \leqslant (1 + \epsilon)\,\mathrm{OPT}$

Algorithm : W_j be jth column of W
D_{W_j} be diagonal matrix with vector W_j

$$W = \begin{bmatrix} 1 & 1 & 1 & \boxed{2} & 2 & 2 \\ 0 & 1 & 2 & \boxed{1} & 2 & 3 \\ 1 & 1 & 3 & \boxed{2} & 4 & 4 \\ 0 & 0 & 4 & \boxed{0} & 4 & 4 \\ 1 & 0 & 5 & \boxed{1} & 6 & 5 \\ 0 & 0 & 6 & \boxed{0} & 6 & 6 \end{bmatrix} \quad D_{W_4} = \begin{bmatrix} 2 & & & & & \\ & 1 & & & & \\ & & 2 & & & \\ & & & 0 & & \\ & & & & 1 & \\ & & & & & 0 \end{bmatrix}$$

Guess a Sketch

Given : $A \in \mathbb{R}^{n\times n}$, $W \in \mathbb{R}^{n\times n}$, $r \in \mathbb{N}$, $k \in \mathbb{N}$, $\epsilon > 0$

Output : $U, V^\top \in \mathbb{R}^{n\times k}$ s.t. $\|W \circ (UV - A)\|_F^2 \leqslant (1+\epsilon)\,\mathrm{OPT}$

Algorithm : W_j be jth column of W

D_{W_j} be diagonal matrix with vector W_j

$$W = \begin{bmatrix} 1 & 1 & 1 & 2 & \boxed{2} & 2 \\ 0 & 1 & 2 & 1 & \boxed{2} & 3 \\ 1 & 1 & 3 & 2 & \boxed{4} & 4 \\ 0 & 0 & 4 & 0 & \boxed{4} & 4 \\ 1 & 0 & 5 & 1 & \boxed{6} & 5 \\ 0 & 0 & 6 & 0 & \boxed{6} & 6 \end{bmatrix} \quad D_{W_5} = \begin{bmatrix} 2 & & & & & \\ & 2 & & & & \\ & & 4 & & & \\ & & & 4 & & \\ & & & & 6 & \\ & & & & & 6 \end{bmatrix}$$

Guess a Sketch

Given : $A \in \mathbb{R}^{n \times n}$, $W \in \mathbb{R}^{n \times n}$, $r \in \mathbb{N}$, $k \in \mathbb{N}$, $\epsilon > 0$

Output : $U, V^\top \in \mathbb{R}^{n \times k}$ s.t. $\| W \circ (UV - A) \|_F^2 \leqslant (1 + \epsilon)\,\mathrm{OPT}$

Algorithm : W_j be jth column of W
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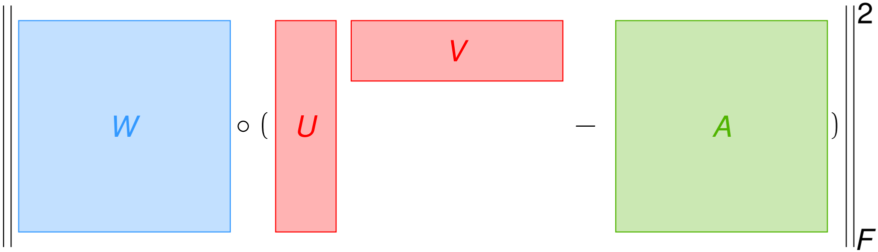
$$W = \begin{bmatrix} 1 & 1 & 1 & 2 & 2 & 2 \\ 0 & 1 & 2 & 1 & 2 & 3 \\ 1 & 1 & 3 & 2 & 4 & 4 \\ 0 & 0 & 4 & 0 & 4 & 4 \\ 1 & 0 & 5 & 1 & 6 & 5 \\ 0 & 0 & 6 & 0 & 6 & 6 \end{bmatrix} \quad D_{W_6} = \begin{bmatrix} 2 & & & & & \\ & 3 & & & & \\ & & 4 & & & \\ & & & 4 & & \\ & & & & 5 & \\ & & & & & 6 \end{bmatrix}$$

Guess a Sketch

Given : $A \in \mathbb{R}^{n \times n}$, $W \in \mathbb{R}^{n \times n}$, $r \in \mathbb{N}$, $k \in \mathbb{N}$, $\epsilon > 0$

Output : $U, V^\top \in \mathbb{R}^{n \times k}$ s.t. $\underbrace{\|W \circ (UV - A)\|_F^2}$ $\leqslant (1+\epsilon)\,\mathrm{OPT}$

Algorithm :

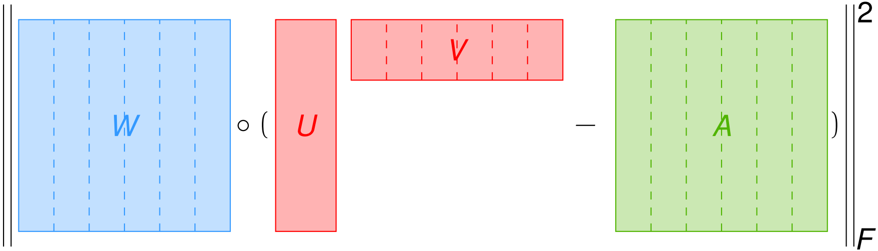


Guess a Sketch

Given : $\quad A \in \mathbb{R}^{n \times n}, W \in \mathbb{R}^{n \times n}, r \in \mathbb{N}, k \in \mathbb{N}, \epsilon > 0$

Output : $\quad U, V^\top \in \mathbb{R}^{n \times k}$ s.t. $\underbrace{\|W \circ (UV - A)\|_F^2}$ $\leqslant (1+\epsilon)\,\mathrm{OPT}$

Algorithm :

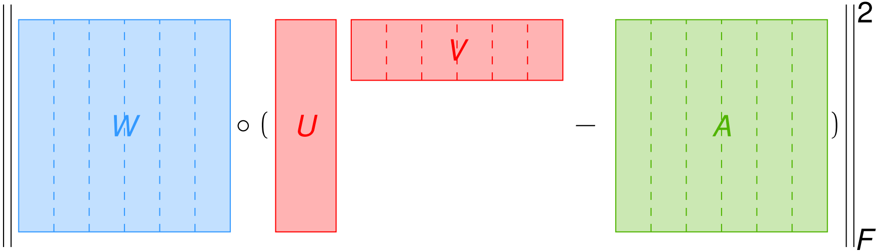


Guess a Sketch

Given : $A \in \mathbb{R}^{n \times n}$, $W \in \mathbb{R}^{n \times n}$, $r \in \mathbb{N}$, $k \in \mathbb{N}$, $\epsilon > 0$

Output : $U, V^{\top} \in \mathbb{R}^{n \times k}$ s.t. $\|W \circ (UV - A)\|_F^2 \leqslant (1 + \epsilon)\,\mathrm{OPT}$

Algorithm : $\sum_{j=1}^{n} \|D_{W_j} U V_j - D_{W_j} A_j\|_2^2$

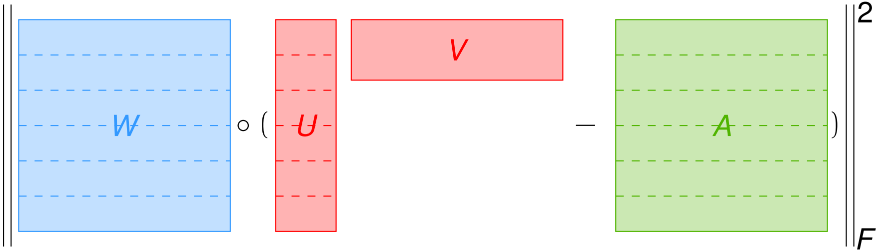


Guess a Sketch

Given : $A \in \mathbb{R}^{n \times n}$, $W \in \mathbb{R}^{n \times n}$, $r \in \mathbb{N}$, $k \in \mathbb{N}$, $\epsilon > 0$

Output : $U, V^\top \in \mathbb{R}^{n \times k}$ s.t. $\|W \circ (UV - A)\|_F^2 \leqslant (1 + \epsilon)\,\mathrm{OPT}$

Algorithm : $\sum_{j=1}^{n} \|D_{W_j} U V_j - D_{W_j} A_j\|_2^2$

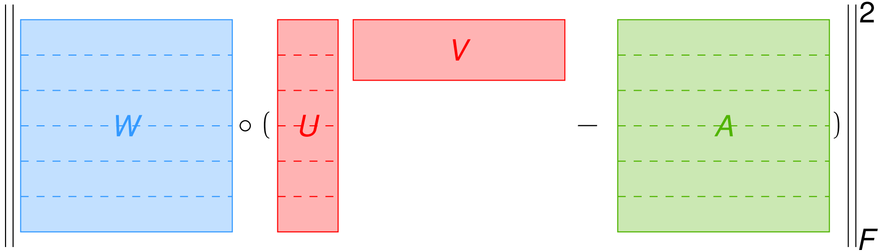


Guess a Sketch

Given : $A \in \mathbb{R}^{n\times n}$, $W \in \mathbb{R}^{n\times n}$, $r \in \mathbb{N}$, $k \in \mathbb{N}$, $\epsilon > 0$

Output : $U, V^\top \in \mathbb{R}^{n\times k}$ s.t. $\underbrace{\| W \circ (UV - A)\|_F^2}$ $\leqslant (1+\epsilon)\,\mathrm{OPT}$

Algorithm : $\sum_{j=1}^{n} \| D_{W_j} U V_j - D_{W_j} A_j \|_2^2 \longleftrightarrow \sum_{i=1}^{n} \| U^i V D_{W^i} - A^i D_{W^i} \|_2^2$



Guess a Sketch

Given : $A \in \mathbb{R}^{n \times n}$, $W \in \mathbb{R}^{n \times n}$, $r \in \mathbb{N}$, $k \in \mathbb{N}$, $\epsilon > 0$

Output : $U, V^\top \in \mathbb{R}^{n \times k}$ s.t. $\|W \circ (UV - A)\|_F^2 \leqslant (1+\epsilon)\,\mathrm{OPT}$

Algorithm : $\sum_{j=1}^{n} \|D_{W_j} U V_j - D_{W_j} A_j\|_2^2 \longleftrightarrow \sum_{i=1}^{n} \|U^i V D_{W^i} - A^i D_{W^i}\|_2^2$

Sketch by Gaussian $S, T^\top \in \mathbb{R}^{t \times n}$

Guess a Sketch

Given : $A \in \mathbb{R}^{n\times n}$, $W \in \mathbb{R}^{n\times n}$, $r \in \mathbb{N}$, $k \in \mathbb{N}$, $\epsilon > 0$

Output : $U, V^\top \in \mathbb{R}^{n\times k}$ s.t. $\underbrace{\|W \circ (UV - A)\|_F^2} \leqslant (1+\epsilon)\,\mathrm{OPT}$

Algorithm : $\sum_{j=1}^{n} \|D_{W_j} U V_j - D_{W_j} A_j\|_2^2 \longleftrightarrow \sum_{i=1}^{n} \|U^i V D_{W^i} - A^i D_{W^i}\|_2^2$

$\downarrow$ Sketch by Gaussian $S, T^\top \in \mathbb{R}^{t\times n}$

$\sum_{j=1}^{n} \|S D_{W_j} U V_j - S D_{W_j} A_j\|_2^2$

Guess a Sketch

Given : $A \in \mathbb{R}^{n\times n}$, $W \in \mathbb{R}^{n\times n}$, $r \in \mathbb{N}$, $k \in \mathbb{N}$, $\epsilon > 0$

Output : $U, V^\top \in \mathbb{R}^{n\times k}$ s.t. $\underbrace{\|W \circ (UV - A)\|_F^2} \leqslant (1+\epsilon)\,\mathrm{OPT}$

Algorithm : $\sum_{j=1}^{n} \|D_{W_j} U V_j - D_{W_j} A_j\|_2^2 \longleftrightarrow \sum_{i=1}^{n} \|U^i V D_{W^i} - A^i D_{W^i}\|_2^2$

Sketch by Gaussian $S, T^\top \in \mathbb{R}^{t\times n}$

$$\sum_{j=1}^{n} \|S D_{W_j} U V_j - S D_{W_j} A_j\|_2^2 \qquad \sum_{i=1}^{n} \|U^i V D_{W^i} T - A^i D_{W^i} T\|_2^2$$

Guess a Sketch

Given : $A \in \mathbb{R}^{n\times n}$, $W \in \mathbb{R}^{n\times n}$, $r \in \mathbb{N}$, $k \in \mathbb{N}$, $\epsilon > 0$

Output : $U, V^\top \in \mathbb{R}^{n\times k}$ s.t. $\underbrace{\|W \circ (UV - A)\|_F^2}$ $\leqslant (1+\epsilon)\,\mathrm{OPT}$

Algorithm : $\sum_{j=1}^{n} \|D_{W_j} U V_j - D_{W_j} A_j\|_2^2 \longleftrightarrow \sum_{i=1}^{n} \|U^i V D_{W^i} - A^i D_{W^i}\|_2^2$

Sketch by Gaussian $S, T^\top \in \mathbb{R}^{t\times n}$

$$\sum_{j=1}^{n} \|S D_{W_j} U V_j - S D_{W_j} A_j\|_2^2 \qquad \sum_{i=1}^{n} \|U^i V D_{W^i} T - A^i D_{W^i} T\|_2^2$$

Guess $S D_{W_j} U \in \mathbb{R}^{t\times k}$ and $V D_{W^i} T \in \mathbb{R}^{k\times t}$

Guess a Sketch

Given : $A \in \mathbb{R}^{n\times n}$, $W \in \mathbb{R}^{n\times n}$, $r \in \mathbb{N}$, $k \in \mathbb{N}$, $\epsilon > 0$

Output : $U, V^\top \in \mathbb{R}^{n\times k}$ s.t. $\|W \circ (UV - A)\|_F^2 \leqslant (1+\epsilon)\,\mathrm{OPT}$

Algorithm : $\sum_{j=1}^{n} \|D_{W_j} U V_j - D_{W_j} A_j\|_2^2 \longleftrightarrow \sum_{i=1}^{n} \|U^i V D_{W^i} - A^i D_{W^i}\|_2^2$

Sketch by Gaussian $S, T^\top \in \mathbb{R}^{t\times n}$

$$\sum_{j=1}^{n} \|S D_{W_j} U V_j - S D_{W_j} A_j\|_2^2 \qquad \sum_{i=1}^{n} \|U^i V D_{W^i} T - A^i D_{W^i} T\|_2^2$$

Guess $S D_{W_j} U \in \mathbb{R}^{t\times k}$ and $V D_{W^i} T \in \mathbb{R}^{k\times t}$

$$S D_{W_1} U = \boxed{S} \begin{bmatrix} 1 & & & & & \\ & 0 & & & & \\ & & 1 & & & \\ & & & 0 & & \\ & & & & 1 & \\ & & & & & 0 \end{bmatrix} \boxed{U}$$

Guess a Sketch

Given : $A \in \mathbb{R}^{n\times n}$, $W \in \mathbb{R}^{n\times n}$, $r \in \mathbb{N}$, $k \in \mathbb{N}$, $\epsilon > 0$

Output : $U, V^\top \in \mathbb{R}^{n\times k}$ s.t. $\|W \circ (UV - A)\|_F^2 \leqslant (1+\epsilon)\,\mathrm{OPT}$

Algorithm : $\sum_{j=1}^{n} \|D_{W_j} U V_j - D_{W_j} A_j\|_2^2 \longleftrightarrow \sum_{i=1}^{n} \|U^i V D_{W^i} - A^i D_{W^i}\|_2^2$

Sketch by Gaussian $S, T^\top \in \mathbb{R}^{t\times n}$

$$\sum_{j=1}^{n} \|S D_{W_j} U V_j - S D_{W_j} A_j\|_2^2 \qquad \sum_{i=1}^{n} \|U^i V D_{W^i} T - A^i D_{W^i} T\|_2^2$$

Guess $SD_{W_j}U \in \mathbb{R}^{t\times k}$ and $VD_{W^i}T \in \mathbb{R}^{k\times t}$

$$SD_{W_2}U = \boxed{S} \begin{bmatrix} 1 &&&&& \\ & 1 &&&& \\ && 1 &&& \\ &&& 0 && \\ &&&& 0 & \\ &&&&& 0 \end{bmatrix} \boxed{U}$$

Guess a Sketch

Given : $A \in \mathbb{R}^{n \times n}$, $W \in \mathbb{R}^{n \times n}$, $r \in \mathbb{N}$, $k \in \mathbb{N}$, $\epsilon > 0$

Output : $U, V^\top \in \mathbb{R}^{n \times k}$ s.t. $\underbrace{\| W \circ (UV - A) \|_F^2} \leqslant (1 + \epsilon)\,\mathrm{OPT}$

Algorithm : $\sum_{j=1}^{n} \| D_{W_j} U V_j - D_{W_j} A_j \|_2^2 \longleftrightarrow \sum_{i=1}^{n} \| U^i V D_{W^i} - A^i D_{W^i} \|_2^2$

Sketch by Gaussian $S, T^\top \in \mathbb{R}^{t \times n}$

$$\sum_{j=1}^{n} \| S D_{W_j} U V_j - S D_{W_j} A_j \|_2^2 \qquad \sum_{i=1}^{n} \| U^i V D_{W^i} T - A^i D_{W^i} T \|_2^2$$

Guess $S D_{W_j} U \in \mathbb{R}^{t \times k}$ and $V D_{W^i} T \in \mathbb{R}^{k \times t}$

$$S D_{W_3} U = \boxed{\;S\;} \begin{bmatrix} 1 & & & & & \\ & 2 & & & & \\ & & 3 & & & \\ & & & 4 & & \\ & & & & 5 & \\ & & & & & 6 \end{bmatrix} \boxed{U}$$

Guess a Sketch

Given : $A \in \mathbb{R}^{n\times n}$, $W \in \mathbb{R}^{n\times n}$, $r \in \mathbb{N}$, $k \in \mathbb{N}$, $\epsilon > 0$

Output : $U, V^\top \in \mathbb{R}^{n\times k}$ s.t. $\underbrace{\|W \circ (UV - A)\|_F^2}$ $\leqslant (1+\epsilon)\,\mathrm{OPT}$

Algorithm : $\sum_{j=1}^{n} \|D_{W_j} U V_j - D_{W_j} A_j\|_2^2 \longleftrightarrow \sum_{i=1}^{n} \|U^i V D_{W^i} - A^i D_{W^i}\|_2^2$

Sketch by Gaussian $S, T^\top \in \mathbb{R}^{t\times n}$

$$\sum_{j=1}^{n} \|S D_{W_j} U V_j - S D_{W_j} A_j\|_2^2 \qquad \sum_{i=1}^{n} \|U^i V D_{W^i} T - A^i D_{W^i} T\|_2^2$$

Guess $S D_{W_j} U \in \mathbb{R}^{t\times k}$ and $V D_{W^i} T \in \mathbb{R}^{k\times t}$

$$S D_{W_4} U = \boxed{\quad S \quad} \begin{bmatrix} 2 &&&&& \\ & 1 &&&& \\ && 2 &&& \\ &&& 0 && \\ &&&& 1 & \\ &&&&& 0 \end{bmatrix} \boxed{U}$$

Guess a Sketch

Given : $A \in \mathbb{R}^{n\times n}$, $W \in \mathbb{R}^{n\times n}$, $r \in \mathbb{N}$, $k \in \mathbb{N}$, $\epsilon > 0$

Output : $U, V^\top \in \mathbb{R}^{n\times k}$ s.t. $\underbrace{\|W \circ (UV - A)\|_F^2}$ $\leqslant (1+\epsilon)\,\mathrm{OPT}$

Algorithm : $\sum_{j=1}^{n} \|D_{W_j} U V_j - D_{W_j} A_j\|_2^2 \longleftrightarrow \sum_{i=1}^{n} \|U^i V D_{W^i} - A^i D_{W^i}\|_2^2$

Sketch by Gaussian $S, T^\top \in \mathbb{R}^{t\times n}$

$$\sum_{j=1}^{n} \|S D_{W_j} U V_j - S D_{W_j} A_j\|_2^2 \qquad \sum_{i=1}^{n} \|U^i V D_{W^i} T - A^i D_{W^i} T\|_2^2$$

Guess $S D_{W_j} U \in \mathbb{R}^{t\times k}$ and $V D_{W^i} T \in \mathbb{R}^{k\times t}$

$$S D_{W_5} U = \boxed{S} \begin{bmatrix} 2 & & & & & \\ & 2 & & & & \\ & & 4 & & & \\ & & & 4 & & \\ & & & & 6 & \\ & & & & & 6 \end{bmatrix} \boxed{U}$$

Guess a Sketch

Given : $A \in \mathbb{R}^{n\times n}$, $W \in \mathbb{R}^{n\times n}$, $r \in \mathbb{N}$, $k \in \mathbb{N}$, $\epsilon > 0$

Output : $U, V^\top \in \mathbb{R}^{n\times k}$ s.t. $\underbrace{\|W \circ (UV - A)\|_F^2}$ $\leqslant (1+\epsilon)\,\mathrm{OPT}$

Algorithm : $\sum_{j=1}^{n} \|D_{W_j} U V_j - D_{W_j} A_j\|_2^2 \longleftrightarrow \sum_{i=1}^{n} \|U^i V D_{W^i} - A^i D_{W^i}\|_2^2$

Sketch by Gaussian $S, T^\top \in \mathbb{R}^{t\times n}$

$\sum_{j=1}^{n} \|S D_{W_j} U V_j - S D_{W_j} A_j\|_2^2$ $\qquad$ $\sum_{i=1}^{n} \|U^i V D_{W^i} T - A^i D_{W^i} T\|_2^2$

Guess $S D_{W_j} U \in \mathbb{R}^{t\times k}$ and $V D_{W^i} T \in \mathbb{R}^{k\times t}$

$$S D_{W_6} U = \boxed{S} \begin{bmatrix} 2 & & & & & \\ & 3 & & & & \\ & & 4 & & & \\ & & & 4 & & \\ & & & & 5 & \\ & & & & & 6 \end{bmatrix} \boxed{U}$$

Guess a Sketch

Given : $A \in \mathbb{R}^{n\times n}$, $W \in \mathbb{R}^{n\times n}$, $r \in \mathbb{N}$, $k \in \mathbb{N}$, $\epsilon > 0$

Output : $U, V^\top \in \mathbb{R}^{n\times k}$ s.t. $\|W \circ (UV - A)\|_F^2 \leqslant (1+\epsilon)\,\mathrm{OPT}$

Algorithm : $\sum_{j=1}^{n} \|D_{W_j} U V_j - D_{W_j} A_j\|_2^2 \longleftrightarrow \sum_{i=1}^{n} \|U^i V D_{W^i} - A^i D_{W^i}\|_2^2$

Sketch by Gaussian $S, T^\top \in \mathbb{R}^{t\times n}$

$$\sum_{j=1}^{n} \|S D_{W_j} U V_j - S D_{W_j} A_j\|_2^2 \qquad \sum_{i=1}^{n} \|U^i V D_{W^i} T - A^i D_{W^i} T\|_2^2$$

Guess $S D_{W_j} U \in \mathbb{R}^{t\times k}$ and $V D_{W^i} T \in \mathbb{R}^{k\times t}$

Create $t \times k$ variables for each of n $S D_{W_j} U$ s?

Guess a Sketch

Given : $A \in \mathbb{R}^{n\times n}$, $W \in \mathbb{R}^{n\times n}$, $r \in \mathbb{N}$, $k \in \mathbb{N}$, $\epsilon > 0$

Output : $U, V^\top \in \mathbb{R}^{n\times k}$ s.t. $\underbrace{\|W \circ (UV - A)\|_F^2}$ $\leqslant (1+\epsilon)\,\mathrm{OPT}$

Algorithm : $\sum_{j=1}^{n} \|D_{W_j} U V_j - D_{W_j} A_j\|_2^2 \longleftrightarrow \sum_{i=1}^{n} \|U^i V D_{W^i} - A^i D_{W^i}\|_2^2$

Sketch by Gaussian $S, T^\top \in \mathbb{R}^{t\times n}$

$$\sum_{j=1}^{n} \|S D_{W_j} U V_j - S D_{W_j} A_j\|_2^2 \qquad \sum_{i=1}^{n} \|U^i V D_{W^i} T - A^i D_{W^i} T\|_2^2$$

Guess $S D_{W_j} U \in \mathbb{R}^{t\times k}$ and $V D_{W^i} T \in \mathbb{R}^{k\times t}$

Create $t \times k$ variables for each of n $S D_{W_j} U$ s?

$$n \times t \times k$$

Guess a Sketch

Given : $A \in \mathbb{R}^{n\times n}$, $W \in \mathbb{R}^{n\times n}$, $r \in \mathbb{N}$, $k \in \mathbb{N}$, $\epsilon > 0$

Output : $U, V^\top \in \mathbb{R}^{n\times k}$ s.t. $\|W \circ (UV - A)\|_F^2 \leqslant (1+\epsilon)\,\mathrm{OPT}$

Algorithm : $\sum_{j=1}^{n} \|D_{W_j} UV_j - D_{W_j} A_j\|_2^2 \longleftrightarrow \sum_{i=1}^{n} \|U^i V D_{W^i} - A^i D_{W^i}\|_2^2$

Sketch by Gaussian $S, T^\top \in \mathbb{R}^{t\times n}$

$$\sum_{j=1}^{n} \|SD_{W_j} UV_j - SD_{W_j} A_j\|_2^2 \qquad \sum_{i=1}^{n} \|U^i V D_{W^i} T - A^i D_{W^i} T\|_2^2$$

Guess $SD_{W_j} U \in \mathbb{R}^{t\times k}$ and $VD_{W^i} T \in \mathbb{R}^{k\times t}$

Create $t \times k$ variables for each of n $SD_{W_j}U$ s?

$$? \times t \times k$$

Guess a Sketch

Given : $A \in \mathbb{R}^{n\times n}$, $W \in \mathbb{R}^{n\times n}$, $r \in \mathbb{N}$, $k \in \mathbb{N}$, $\epsilon > 0$

Output : $U, V^\top \in \mathbb{R}^{n\times k}$ s.t. $\underbrace{\|W \circ (UV - A)\|_F^2}$ $\leqslant (1+\epsilon)\,\mathrm{OPT}$

Algorithm : $\sum_{j=1}^{n} \|D_{W_j} U V_j - D_{W_j} A_j\|_2^2 \longleftrightarrow \sum_{i=1}^{n} \|U^i V D_{W^i} - A^i D_{W^i}\|_2^2$

Sketch by Gaussian $S, T^\top \in \mathbb{R}^{t\times n}$

$$\sum_{j=1}^{n} \|S D_{W_j} U V_j - S D_{W_j} A_j\|_2^2 \qquad \sum_{i=1}^{n} \|U^i V D_{W^i} T - A^i D_{W^i} T\|_2^2$$

Guess $SD_{W_j}U \in \mathbb{R}^{t\times k}$ and $VD_{W^i}T \in \mathbb{R}^{k\times t}$

Create $t \times k$ variables for each of n $SD_{W_j}U$ s?

$$r \times t \times k$$

Guess a Sketch

Given : $A \in \mathbb{R}^{n \times n}$, $W \in \mathbb{R}^{n \times n}$, $r \in \mathbb{N}$, $k \in \mathbb{N}$, $\epsilon > 0$

Output : $U, V^\top \in \mathbb{R}^{n \times k}$ s.t. $\| W \circ (UV - A) \|_F^2 \leqslant (1+\epsilon)\,\mathrm{OPT}$

Algorithm : W_j be jth column of W

$$W = \begin{bmatrix} 1 & 1 & 1 & 2 & 2 & 2 \\ 0 & 1 & 2 & 1 & 2 & 3 \\ 1 & 1 & 3 & 2 & 4 & 4 \\ 0 & 0 & 4 & 0 & 4 & 4 \\ 1 & 0 & 5 & 1 & 6 & 5 \\ 0 & 0 & 6 & 0 & 6 & 6 \end{bmatrix}$$

Guess a Sketch

Given : $A \in \mathbb{R}^{n\times n}$, $W \in \mathbb{R}^{n\times n}$, $r \in \mathbb{N}$, $k \in \mathbb{N}$, $\epsilon > 0$

Output : U, $V^\top \in \mathbb{R}^{n\times k}$ s.t. $\|W \circ (UV - A)\|_F^2 \leqslant (1+\epsilon)\,\mathrm{OPT}$

Algorithm : W_j be jth column of W

$$W = \begin{bmatrix} 1 & 1 & 1 & 2 & 2 & 2 \\ 0 & 1 & 2 & 1 & 2 & 3 \\ 1 & 1 & 3 & 2 & 4 & 4 \\ 0 & 0 & 4 & 0 & 4 & 4 \\ 1 & 0 & 5 & 1 & 6 & 5 \\ 0 & 0 & 6 & 0 & 6 & 6 \end{bmatrix} \qquad \overbrace{\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 1 & 3 \\ 0 & 0 & 4 \\ 1 & 0 & 5 \\ 0 & 0 & 6 \end{bmatrix}}^{\text{column span of } W}$$

Guess a Sketch

Given : $A \in \mathbb{R}^{n\times n}$, $W \in \mathbb{R}^{n\times n}$, $r \in \mathbb{N}$, $k \in \mathbb{N}$, $\epsilon > 0$

Output : $U, V^\top \in \mathbb{R}^{n\times k}$ s.t. $\|W \circ (UV - A)\|_F^2 \leqslant (1+\epsilon)\,\mathrm{OPT}$

Algorithm : W_j be jth column of W

$$W_1 = W_1$$

$$W = \begin{bmatrix} \boxed{\begin{matrix}1\\0\\1\\0\\1\\0\end{matrix}} & \begin{matrix}1\\1\\1\\0\\0\\0\end{matrix} & \begin{matrix}1\\2\\3\\4\\5\\6\end{matrix} & \begin{matrix}2\\1\\2\\0\\1\\0\end{matrix} & \begin{matrix}2\\2\\4\\4\\6\\6\end{matrix} & \begin{matrix}2\\3\\4\\4\\5\\6\end{matrix} \end{bmatrix} \qquad \begin{bmatrix}1\\0\\1\\0\\1\\0\end{bmatrix} = \begin{bmatrix}1\\0\\1\\0\\1\\0\end{bmatrix}$$

Guess a Sketch

Given : $A \in \mathbb{R}^{n\times n}$, $W \in \mathbb{R}^{n\times n}$, $r \in \mathbb{N}$, $k \in \mathbb{N}$, $\epsilon > 0$

Output : $U, V^\top \in \mathbb{R}^{n\times k}$ s.t. $\|W \circ (UV - A)\|_F^2 \leqslant (1+\epsilon)\,\mathrm{OPT}$

Algorithm : W_j be jth column of W

$$W_2 = W_2$$

$$W = \begin{bmatrix} 1 & \boxed{1} & 1 & 2 & 2 & 2 \\ 0 & 1 & 2 & 1 & 2 & 3 \\ 1 & 1 & 3 & 2 & 4 & 4 \\ 0 & 0 & 4 & 0 & 4 & 4 \\ 1 & 0 & 5 & 1 & 6 & 5 \\ 0 & 0 & 6 & 0 & 6 & 6 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Guess a Sketch

Given : $A \in \mathbb{R}^{n\times n}$, $W \in \mathbb{R}^{n\times n}$, $r \in \mathbb{N}$, $k \in \mathbb{N}$, $\epsilon > 0$

Output : $U, V^\top \in \mathbb{R}^{n\times k}$ s.t. $\|W \circ (UV - A)\|_F^2 \leqslant (1+\epsilon)\,\mathrm{OPT}$

Algorithm : W_j be jth column of W

$$W_3 = W_3$$

$$W = \begin{bmatrix} 1 & 1 & 1 & 2 & 2 & 2 \\ 0 & 1 & 2 & 1 & 2 & 3 \\ 1 & 1 & 3 & 2 & 4 & 4 \\ 0 & 0 & 4 & 0 & 4 & 4 \\ 1 & 0 & 5 & 1 & 6 & 5 \\ 0 & 0 & 6 & 0 & 6 & 6 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{bmatrix}$$

Guess a Sketch

Given : $A \in \mathbb{R}^{n\times n}$, $W \in \mathbb{R}^{n\times n}$, $r \in \mathbb{N}$, $k \in \mathbb{N}$, $\epsilon > 0$

Output : U, $V^\top \in \mathbb{R}^{n\times k}$ s.t. $\|W \circ (UV - A)\|_F^2 \leqslant (1+\epsilon)\,\mathrm{OPT}$

Algorithm : W_j be jth column of W

$$W_4 = W_1 + W_2$$

$$W = \begin{bmatrix} 1 & 1 & 1 & 2 & 2 & 2 \\ 0 & 1 & 2 & 1 & 2 & 3 \\ 1 & 1 & 3 & 2 & 4 & 4 \\ 0 & 0 & 4 & 0 & 4 & 4 \\ 1 & 0 & 5 & 1 & 6 & 5 \\ 0 & 0 & 6 & 0 & 6 & 6 \end{bmatrix} \qquad \begin{bmatrix} 2 \\ 1 \\ 2 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Guess a Sketch

Given : $A \in \mathbb{R}^{n \times n}$, $W \in \mathbb{R}^{n \times n}$, $r \in \mathbb{N}$, $k \in \mathbb{N}$, $\epsilon > 0$

Output : $U, V^\top \in \mathbb{R}^{n \times k}$ s.t. $\| W \circ (UV - A) \|_F^2 \leqslant (1+\epsilon)\,\mathrm{OPT}$

Algorithm : W_j be jth column of W

$$W_5 = W_1 + W_3$$

$$W = \begin{bmatrix} 1 & 1 & 1 & 2 & \boxed{2} & 2 \\ 0 & 1 & 2 & 1 & \boxed{2} & 3 \\ 1 & 1 & 3 & 2 & \boxed{4} & 4 \\ 0 & 0 & 4 & 0 & \boxed{4} & 4 \\ 1 & 0 & 5 & 1 & \boxed{6} & 5 \\ 0 & 0 & 6 & 0 & \boxed{6} & 6 \end{bmatrix} \qquad \begin{bmatrix} 2 \\ 2 \\ 4 \\ 4 \\ 6 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{bmatrix}$$

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create $t \times k$ variables for $S D_{W_1} U$

$$S \qquad \begin{bmatrix} 1 & & & & & \\ & 0 & & & & \\ & & 1 & & & \\ & & & 0 & & \\ & & & & 1 & \\ & & & & & 0 \end{bmatrix} \qquad U$$

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$$S \qquad \begin{bmatrix} 2 & & & & & \\ & 1 & & & & \\ & & 2 & & & \\ & & & 0 & & \\ & & & & 1 & \\ & & & & & 0 \end{bmatrix} \qquad U$$

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$$S \begin{bmatrix} 1+1 & & & & & \\ & 0+1 & & & & \\ & & 1+1 & & & \\ & & & 0+0 & & \\ & & & & 1+0 & \\ & & & & & 0+0 \end{bmatrix} U$$

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- Time : $n^{O(rk^2/\epsilon)}$

$$(\#\,\text{constraints} \cdot \text{degree})^{O(\#\,\text{variables})} = (O(n) \cdot O(nk))^{O(krt)}$$

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- Can we prove a hardness result with respect to the parameter k, e.g., a $2^{\Omega(k)}$ lower bound for WLRA problem?

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  - nonnegative matrix factorization
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- Hardness techniques
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- Parameterized Complexity gives a way of coping with intractability for emerging machine learning problems

Topics at a Survey Level

- Projection onto Complicated Objects and Gaussian Mean Width
- M-Estimator Loss Functions for Regression
- Compressed Sensing

Projection onto other Objects

- Least squares regression finds the closest point y in a subspace K to a given point b
- Given a (possibly infinite) set of points K, and a point b, compute $\min_{y \in K} |y - b|$
  - All norms are Euclidean norms
- Let S be a sketching matrix, we want that if y' = $\mathrm{argmin}_{y \in K} |Sy - Sb|$, then

$$|y' - b| \leq (1 + \epsilon) \min_{y \in K} |y - b|$$

- More generally, want to preserve distances of all vectors in a set K, that is, |S(y-y')| = $(1 \pm \epsilon)|y - y'|$ for all y, y' ∈ K

What properties of K determine the dimension and sparsity of S?

Example: Preserving Distances in a Set

- More generally, want to preserve distances of all vectors in a set K, that is, |S(y-y')| = $(1 \pm \epsilon)|y - y'|$ for all y, y' ∈ K

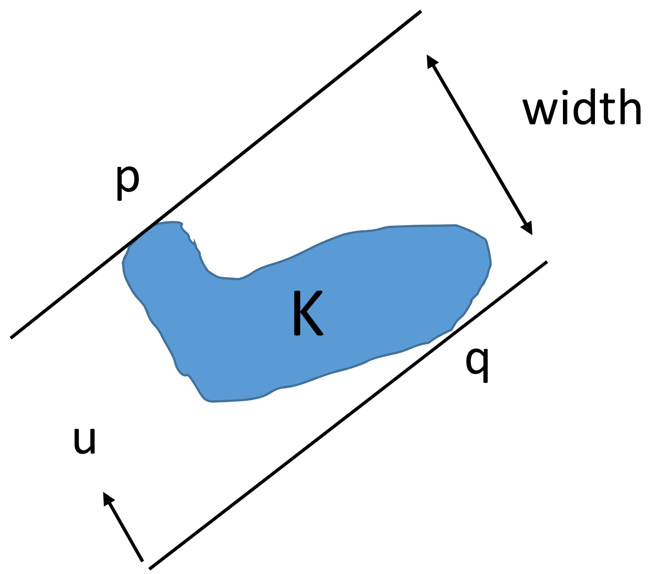
- What is the dimension of S needed if K is:

  - n arbitrary points in R^{d}?

  - n arbitrary points on a line in R^{d}?

Spherical Mean Width

- Let K be a bounded subset in R^n
- Consider the width in direction u for a unit vector u:



- Width in direction u = $\sup_{\mathrm{p,q\ in\ K}} < \mathrm{u}, \mathrm{p} - \mathrm{q} >$
- Spherical mean width = $\mathrm{E_u}[\sup_{\mathrm{p,q\ in\ K}} < \mathrm{u}, \mathrm{p} - \mathrm{q} >]$

Gaussian Mean Width

- Let $g \sim N(0, I_n)$ be an i.i.d. Gaussian vector

- Gaussian mean width g(K) = $E_g[\sup_{p,q \text{ in } K} < g, p - q >]$

  $= \Theta(n^{.5}) \cdot$ spherical mean width

- Examples

  - K = S^{n-1}
    - $\Theta(n^{.5})$
  - K = set of unit vectors in a d-dimensional subspace of R^n
    - $\Theta(d^{.5})$
  - K = t arbitrary unit vectors in R^n
    - $\Theta(\log^{.5} t)$

Gaussian Mean Width of t Arbitrary Unit Vectors

- Let $u^1, \ldots, u^t$ be t arbitrary unit vectors in R^n
- Let g in R^n have iid N(0,1) entries
- Define random variables $Z_j = < u^j, g >$ which are N(0,1) random variables
- Want to bound $E_g[\max_j Z_j]$
- Fact: for an N(0,1) random variable W, $E\left[e^{\lambda W}\right] = e^{\lambda^2/2}$
- For any $\lambda > 0$, $E\left[e^{\lambda \max_j Z_j}\right] \leq \sum_j E\left[e^{\lambda Z_j}\right] \leq t\, e^{\lambda^2/2}$
- For all $\lambda > 0$, $E_g[\max_j Z_j] \leq \left(\frac{1}{\lambda}\right) \log E[e^{\lambda \max_j Z_j}] \leq (\frac{\log t}{\lambda} + \frac{\lambda}{2}) = 2\sqrt{\log t}$

Sketching Bounds

- [Gordon] Let K be a subset of S^{n-1}. A random Gaussian matrix S with $g(K)^2/\epsilon^2$ rows satisfies
$$|S(y-y')|^2 = (1 \pm \epsilon)|y-y'|^2 \text{ for all y, y' in K}$$

- What about sparse sketching matrices S?

- [Bourgain, Dirksen, Nelson] S can have m = $g(K)^2 \text{poly}(\log n)/\epsilon^2$ rows and $s = \text{poly}(\log n)/\epsilon^2$ non- zeros per column if m and s satisfy a condition related to higher moments of $\sup_{p,q} < g, p-q >$
  - Applied to finite and infinite unions of subspaces

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Other Fitness Measures

Example: Method of least absolute deviation (l_1 -regression)

- Find x* that minimizes $|Ax-b|_1 = \Sigma\ |b_i - \langle A_{i*}, x\rangle|$
- Cost is less sensitive to outliers than least squares
- Can solve via linear programming

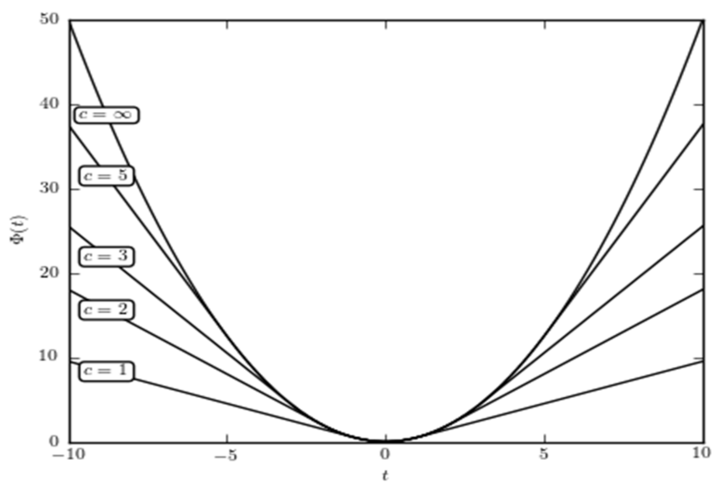
What about the many other fitness measures used in practice?

Huber Loss Function

$M(x) = x^2/(2c)$ for $|x| \leq c$

$M(x) = |x|-c/2$ for $|x| > c$

Enjoys smoothness properties of l_2^2 and robustness properties of l_1



Other Examples

- L_1-L_2

  $M(x) = 2((1+x^2/2)^{1/2} - 1)$

- Fair estimator

  $M(x) = c^2 [\ |x|/c - \log(1+|x|/c)\ ]$

- Tukey estimator

  $M(x) = c^2/6\ (1-[1-(x/c)^2]^3) \quad \text{if } |x| \leq c$

  $\quad = c^2/6 \quad \text{if } |x| > c$

Nice M-Estimators

- An M-Estimator is nice if it has at least linear growth and at most quadratic growth

- There is $C_M > 0$ so that for all a, a’ with $|a| \geq |a'| > 0$,

  $$|a/a'|^2 \geq M(a)/M(a') \geq C_M\, |a/a'|$$

- Any convex M satisfies the linear lower bound

  $$M(a') = M\left(\frac{a'}{a} \cdot a + \left(1 - \frac{a'}{a}\right) \cdot 0\right) \leq \frac{a'}{a} M(a) + \left(1 - \frac{a'}{a}\right) M(0) = \frac{a'}{a} M(a)$$

- Any sketchable M satisfies the quadratic upper bound
  - sketchable => there is a distribution on k x n matrices S for which $|Sx|_M = \Theta(|x|_M)$ with good probability and k is slow-growing function of n

Nice M-Estimator Theorem

[Nice M-Estimators] $O(nnz(A)) + poly(d \log n)$ time algorithm to output x'
so that for any constant $C > 1$, with probability 99%:

$$|Ax'-b|_M \leq C \min_x |Ax-b|_M$$

Remarks:

- For convex nice M-estimators can solve with convex programming, but slow – poly(nd) time

- The sketch is "universal"

M-Sketch

$$T = \begin{bmatrix} S^0 \cdot D^0 \\ S^1 \cdot D^1 \\ S^2 \cdot D^2 \\ \cdots \\ S^{\log n} \cdot D^{\log n} \end{bmatrix}$$

-The same M-Sketch works for all nice M-estimators!

$x' = \text{argmin}_x \ |TAx\text{-}Tb|_{w,M}$

- S^i are independent CountSketch matrices with poly(d) rows
- D^i is n x n diagonal and uniformly samples a $1/(d \log n)^i$ fraction of the n rows
- Want $|T(Ax - b)|_{w,M} \approx |Ax - b|_M$ for any given y = Ax-b
  – Example: y = (n, 1, …, 1): $|y|_1 = n + 1 \cdot (n - 1)$. Estimate both scales

Topics at a Survey Level

- Projection onto Complicated Objects and Gaussian Mean Width
- M-Estimator Loss Functions for Regression
- Compressed Sensing

Compressed Sensing

- We take random “linear measurements” of an n-dimensional vector x
- In our language, we choose a random r x n sketching matrix S and observe $\mathrm{S} \cdot \mathrm{x}$
- Output a vector x’ with $|\mathrm{x}-\mathrm{x}'|_{\mathrm{p}} = \mathrm{D} \cdot \min_{\mathrm{k-sparse\ z}} |\mathrm{x}-\mathrm{z}|_{\mathrm{q}}$, where D is the distortion (the $\ell_{\mathrm{p}}/\ell_{\mathrm{q}}$-guarantee)
- Let x_{k} be the best k-sparse approximation to x, i.e., the largest k coordinates in magnitude
- Randomized (“for-each”) scheme versus deterministic (“for-all”) scheme
- CountSketch is a randomized scheme achieving ℓ_2/ℓ_2 w.h.p.

$$|\mathrm{x}-\mathrm{x}'|_2 = \mathrm{O}(1)\cdot |\mathrm{x}-\mathrm{x}_{\mathrm{k}}|_2$$

CountSketch for Compressed Sensing

- CountSketch had O(log n) repetitions of hashing into O(k) buckets
- S is a random linear map S with O(k log n) rows
- For an n-dimensional vector x, estimate every x_i up to additive error $\frac{|x-x_k|_2}{\sqrt{k}}$
- Output a 2k-sparse x' consisting of the top 2k estimates given by CountSketch
- Say coordinate i is heavy if $|x_i| \geq |x - x_k|_2/\sqrt{k}$
  - How many heavy coordinates can there be?
- Say a coordinate i is super-heavy if $|x_i| \geq 3|x - x_k|_2/\sqrt{k}$
  - Claim: the set T of super-heavy coordinates is in the support of x'
- $|x - x'|_2 \leq |(x - x')_T|_2 + \left|(x - x')_{[n]\setminus T}\right|_2$

$$\leq \sqrt{2k} \cdot \frac{|x-x_k|_2}{\sqrt{k}} + |(x - x_k)_{[n]\setminus T}|_2 + \left|(x_k - x')_{[n]\setminus T}\right|_2 = O(|x - x_k|_2)$$

No Deterministic Algorithm Achieves ℓ_2/ℓ_2

- Recall ℓ_2/ℓ_2: output x' with $|x - x'|_2 = O(1) \cdot |x - x_k|_2$
- Consider k = 1
- Suppose S is a deterministic sketching matrix with r = o(n) rows
- Suffices to show there is a vector x in kernel(S) with $|x|_\infty \geq C|x - x_1|_2$ for any constant C > 0
- W.l.o.g., can assume S has orthonormal rows
- $\sum_i |Se_i|_2^2 = r$, so there exists an i with $|Se_i|_2^2 \leq \frac{r}{n}$
- Let $x = e_i - S^T S e_i$, so x is in kernel(S)
- But $|x|_\infty^2 \geq |x_i|^2 = \left(e_i^T e_i - e_i^T S^T S e_i\right)^2 \geq \left(1 - \frac{r}{n}\right)^2$, while
- $|x - x_1|_2 \leq |x - e_i|_2 = \left|S^T S e_i\right|_2 = |Se_i|_2 \leq \sqrt{\frac{r}{n}} = o(1)$

Deterministic Algorithms Achieve ℓ_2/ℓ_1

- ℓ_2/ℓ_1: output x' with $|x - x'|_2 = O(1/k^{.5}) \cdot |x - x_k|_1$
- S has the (ϵ, k)-restricted isometry property (RIP) if for all k-sparse vectors x,

$$(1-\epsilon)|x|_2^2 \le |Sx|_2^2 \le (1+\epsilon)|x|_2^2$$

- What are some matrices S with O(k log(n/k)) rows that have the (ϵ, k)-RIP property for constant ϵ?
- Deterministic, but not explicit!
- Major open question: explicit matrix with (ϵ, k)-RIP with $o(k^2)$ rows
- Bourgain et al.: can get $k^{2-\gamma}$ rows for a constant $\gamma > 0$ and $k \approx n^{.5}$

Deterministic Algorithms Achieve ℓ_2/ℓ_1

- If S has the (ϵ, k)-RIP then one can efficiently output an x' for which
$$|\mathrm{x} - \mathrm{x}'|_2 = \mathrm{O}(1/\mathrm{k}^{.5}) \cdot |\mathrm{x} - \mathrm{x}_\mathrm{k}|_1$$

- In fact, can just solve a linear program!

$$\min_{\mathrm{z} \in \mathrm{R}^\mathrm{n}} |\mathrm{z}|_1$$
s.t. Sz = Sx

- If x' is the solution, then $|\mathrm{x} - \mathrm{x}'|_2 \leq \mathrm{O}\left(\frac{1}{\mathrm{k}^{.5}}\right) |\mathrm{x} - \mathrm{x}_\mathrm{k}|_1$

- Proof uses (ϵ, k)-RIP and elementary norm manipulations