15-853:Algorithms in the Real World

Cryptography 1 and 2

Cryptography Outline

Introduction: terminology, cryptanalysis, security

Primitives: one-way functions, trapdoors, …

Protocols: digital signatures, key exchange, ..

Number Theory: groups, fields, …

Private-Key Algorithms: Rijndael, DES

Public-Key Algorithms: Knapsack, RSA, El-Gamal, …

Case Studies: Kerberos, SSL

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- cryptanalytic attacks
- security

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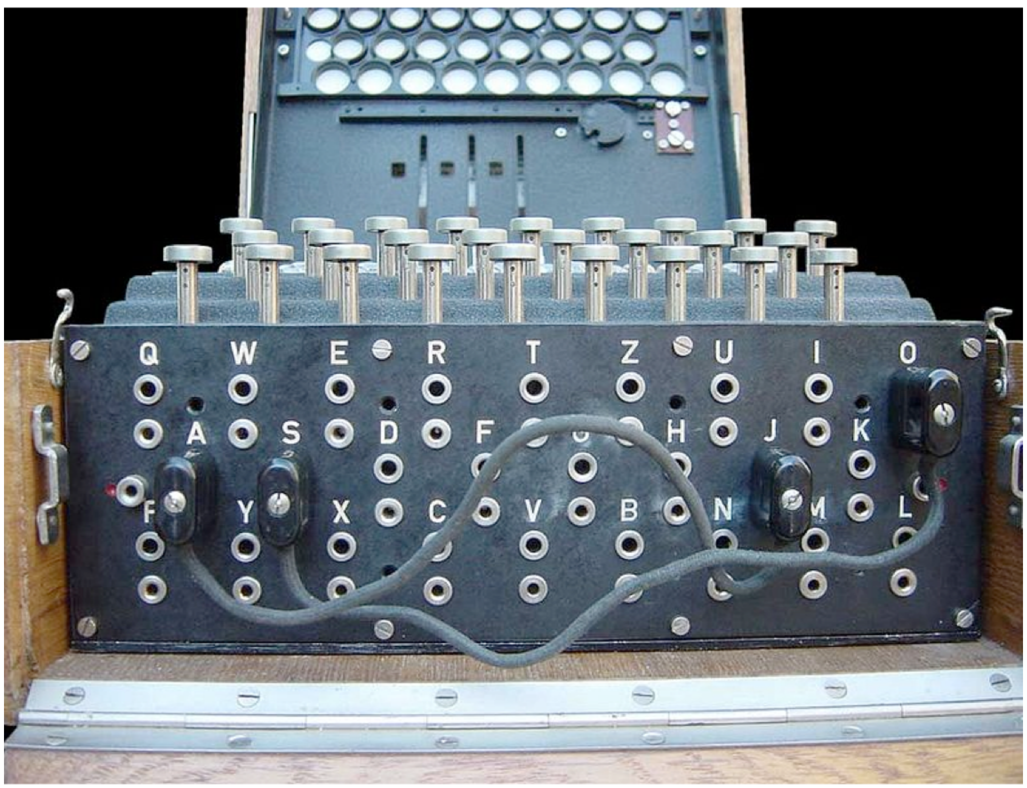
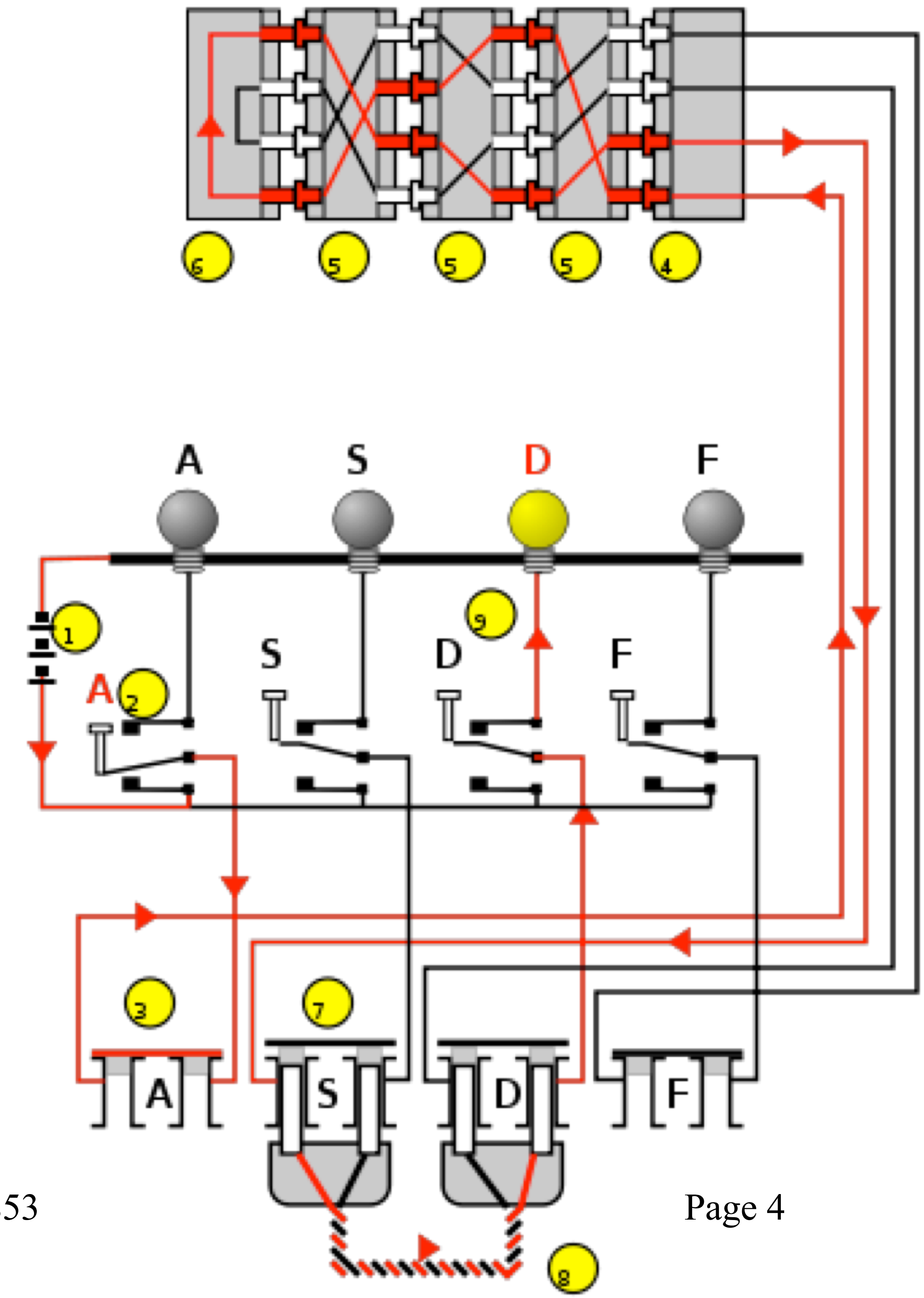
Enigma Machine

"It was thanks to Ultra
that we won the war."
- Winston Churchill





Some Terminology

Cryptography – the general term

Cryptology – the mathematics

Encryption – encoding but sometimes used as general term)

Cryptanalysis – breaking codes

Steganography – hiding message

Cipher – a method or algorithm for encrypting or decrypting

More Definitions

Plaintext

Key_1 → **Encryption** $E_k(M) = C$

Cyphertext

Key_2 → **Decryption** $D_k(C) = M$

Original Plaintext

Private Key or **Symmetric**: $Key_1 = Key_2$

Public Key or **Asymmetric**: $Key_1 \neq Key_2$

Key_1 or Key_2 is public depending on the protocol

Cryptanalytic Attacks

C = ciphertext messages

M = plaintext messages

Ciphertext Only: Attacker has multiple **C**s but does not know the corresponding **M**s

Known Plaintext: Attacker knows some number of (**C**,**M**) pairs.

Chosen Plaintext: Attacker gets to choose **M** and generate **C**.

Chosen Ciphertext: Attacker gets to choose **C** and generate **M**.

What does it mean to be secure?

Unconditionally Secure: Encrypted message cannot be decoded without the key

Shannon showed in 1943 that key must be as long as the message to be unconditionally secure – this is based on information theory

A **one time pad** – xor a random key with a message (Used in 2nd world war)

Security based on computational cost: it is computationally "infeasible" to decode a message without the key.

No (probabilistic) polynomial time algorithm can decode the message.

The Cast

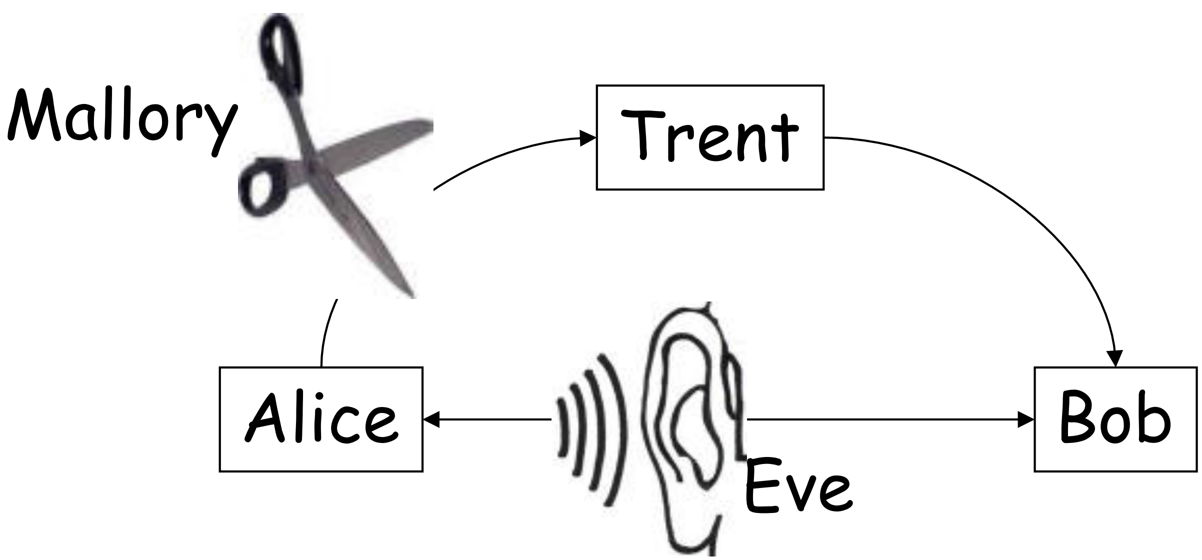
Alice – initiates a message or protocol

Bob - second participant

Trent – trusted middleman

Eve – eavesdropper

Mallory – malicious active attacker



Cryptography Outline

Introduction: terminology, cryptanalysis, security

Primitives:

- one-way functions
- one-way trapdoor functions
- one-way hash functions

Protocols: digital signatures, key exchange, ..

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Primitives: One-Way Functions

(Informally): A function

$y = f(x)$

*is **one-way** if it is easy to compute **y** from **x** but "hard" to compute **x** from **y***

Building block of most cryptographic protocols

And, the security of most protocols rely on their existence.

Unfortunately, not known to exist. This is true even if we assume $P \neq NP$.

One-way functions: possible definition

1. $F(x)$ is polynomial time
2. $F^{-1}(x)$ is NP-hard

What is wrong with this definition?

One-way functions: better definition

For most **y** no single PPT (probabilistic polynomial time) algorithm can compute **x**

Roughly: at most a fraction $1/|x|^k$ instances **x** are easy for any **k** and as $|x| \to \infty$

This definition can be used to make the probability of hitting an easy instance arbitrarily small.

There are nice results on "pumping" to increase k.

One-way functions: better definition

Also important that cannot get any information about $y = F^{-1}(x)$. Even getting 1 bit of y, or some slightly skewed probability over y could be dangerous in a encryption scheme.

Some examples (conjectures)

Factoring:

$x = (u,v)$

$y = f(u,v) = u*v$

If u and v are prime it is hard to generate them from y.

Discrete Log: $y = g^x \bmod p$

where p is prime and g is a "generator" (*i.e.*, g^1, g^2, g^3, … generates all values < p).

DES with fixed message: $y = DES_x(m)$

This would assume a family of DES functions of increasing key size (for asymptotics)

One-way functions in private-key protocols

y = ciphertext m = plaintext k = key

Consider

$y = E_k(m) = E(k,m) = E_m(k)$ (i.e. $f = E_m$)

should this be a one-way function?

In a **known-plaintext attack** we know a (y,m) pair.

The m along with E defines f

$E_m(k)$ needs to be easy

$E_m^{-1}(y)$ should be hard

Otherwise we could extract the key k.

One-Way Trapdoor Functions

*A **one-way** function with a "trapdoor"*

*The **trapdoor** is a key that makes it easy to invert the function $y = f(x)$*

Example: **RSA** (conjecture)

- $y = x^e \bmod n$
- Where $n = pq$ (p, q, e are prime)
- p or q or d (where $ed = (p-1)(q-1) \bmod n$) can be used as trapdoors

In public-key algorithms

- f(x) = public key (*e.g.*, e and n in RSA)
- Trapdoor = private key (*e.g.*, d in RSA)

One-way Hash Functions

Y = h(x) where

- y is a fixed length independent of the size of x. In general this means h is not invertible since it is many to one.
- Calculating y from x is easy
- Calculating <u>any</u> x such that y = h(x) give y is hard

Used in digital signatures and other protocols.

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Protocols

Other protocols:

- Authentication
- Secret sharing
- Timestamping services
- Zero-knowledge proofs
- Blind-signatures
- Key-escrow
- Secure elections
- Digital cash

Implementation of the protocol is often the weakest point in a security system.

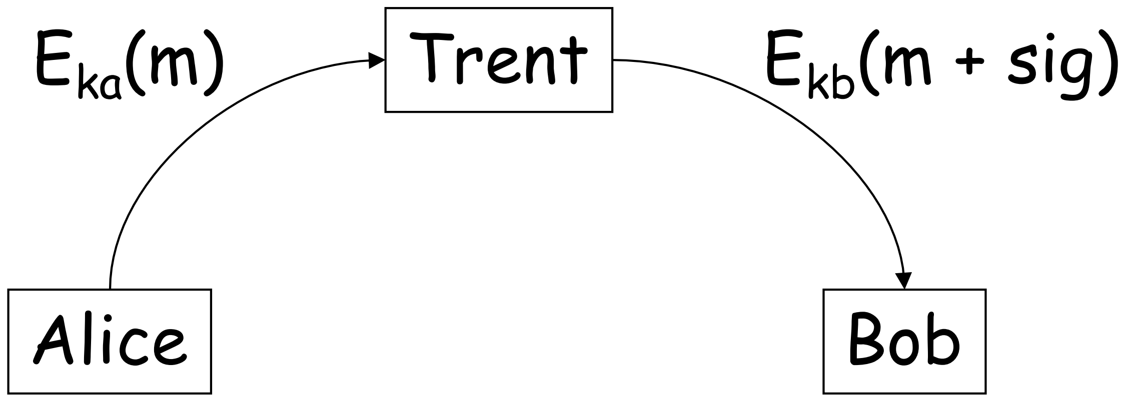
Protocols: Digital Signatures

Goals:

1. Convince recipient that message was actually sent by a trusted source
2. Do not allow repudiation, *i.e.,* that's not my signature.
3. Do not allow tampering with the message without invalidating the signature

Item 2 turns out to be really hard to do

Using private keys



- **ka** is a secret key shared by Alice and Trent
- **kb** is a secret key shared by Bob and Trent

sig is a note from Trent saying that Alice “signed” it.

To prevent repudiation Trent needs to keep m or at least h(m) in a database

Using Public Keys

$$\boxed{\text{Alice}} \xrightarrow{D_{k1}(m)} \boxed{\text{Bob}}$$

K1 = Alice's private key
Bob decrypts it with her public key

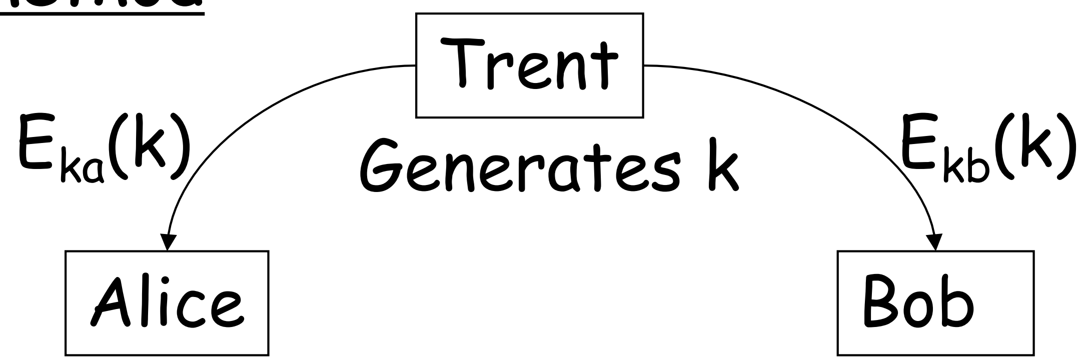
More Efficiently

$$\boxed{\text{Alice}} \xrightarrow{D_{k1}(h(m)) + m} \boxed{\text{Bob}}$$

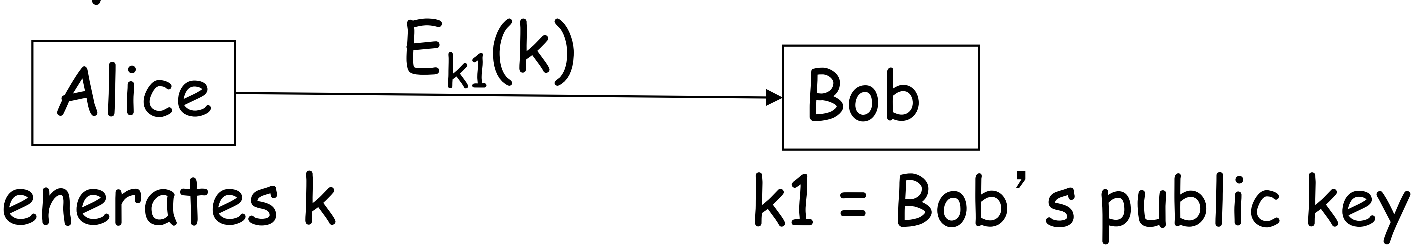
h(m) is a one-way hash of m

Key Exchange

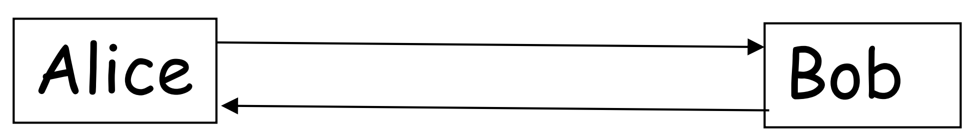
Private Key method



Public Key method



Key exchange protocol (e.g. Diffie Hellman)



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Number Theory Review: (Mostly covered last week)

- Groups
- Fields
- Polynomials and Galois fields

Private-Key Algorithms: Rijndael, DES

Public-Key Algorithms: Knapsack, RSA, El-Gamal, …

Case Studies: Kerberos, Digital Cash

Number Theory Outline

Groups

- Definitions, Examples, Properties
- Multiplicative group modulo n
- The Euler-phi function

Fields

- Definition, Examples
- Polynomials
- Galois Fields

Why does number theory play such an important role?

It is **the** mathematics of finite sets of values.

Groups

A **Group** $(G,*,I)$ is a set G with operator $*$ such that:

1. **Closure.** For all $a,b \in G$, $a * b \in G$
2. **Associativity.** For all $a,b,c \in G$, $a*(b*c) = (a*b)*c$
3. **Identity.** There exists $I \in G$, such that for all $a \in G$, $a*I=I*a=a$
4. **Inverse.** For every $a \in G$, there exist a unique element $b \in G$, such that $a*b=b*a=I$

An **Abelian or Commutative Group** is a Group with the additional condition

5. **Commutativity.** For all $a,b \in G$, $a*b=b*a$

Examples of groups

- Integers, Reals or Rationals with Addition
- The nonzero Reals or Rationals with Multiplication
- Non-singular n x n real matrices with Matrix Multiplication
- Permutations over n elements with composition
  $[0\to1, 1\to2, 2\to0] \circ [0\to1, 1\to0, 2\to2] = [0\to0, 1\to2, 2\to1]$

We will only be concerned with **finite groups**, I.e., ones with a finite number of elements.

Key properties of finite groups

Notation: $a^j \equiv a * a * a * \ldots$ j times

Theorem (Fermat's little): for any finite group $(G,*,I)$ and $g \in G$, $g^{|G|} = I$

Definition: the **order** of $g \in G$ is the smallest positive integer m such that $g^m = I$

Definition: a group G is **cyclic** if there is a $g \in G$ such that order(g) = |G|

Definition: an element $g \in G$ of order |G| is called a **generator** or **primitive element** of G.

Groups based on modular arithmetic

The group of positive integers modulo a prime p

$Z_p^* \equiv \{1, 2, 3, \ldots, p-1\}$

$*_p \equiv$ multiplication modulo p

Denoted as: $(Z_p^*, *_p)$

Required properties

1. Closure. Yes.
2. Associativity. Yes.
3. Identity. 1.
4. Inverse. Yes.

Example: $Z_7^* = \{1,2,3,4,5,6\}$

$1^{-1} = 1,\ 2^{-1} = 4,\ 3^{-1} = 5,\ 6^{-1} = 6$

Other properties

$|Z_p^*| = (p-1)$

By Fermat's little theorem: $a^{(p-1)} = 1 \pmod p$

Example of Z_7^*

Generators

x	x^2	x^3	x^4	x^5	x^6
1	1	1	1	1	1
2	4	1	2	4	1
3	**2**	**6**	**4**	**5**	**1**
4	2	1	4	2	1
5	**4**	**6**	**2**	**3**	**1**
6	1	6	1	6	1

For all p the group is cyclic.

What if n is not a prime?

The group of positive integers modulo a non-prime n

$Z_n \equiv \{1, 2, 3, \ldots, n\text{-}1\}$, n not prime

$*_p \equiv$ multiplication modulo n

Required properties?

1. Closure. ?
2. Associativity. ?
3. Identity. ?
4. Inverse. ?

How do we fix this?

Groups based on modular arithmetic

The **multiplicative group modulo n**

$Z_n^* \equiv \{m : 1 \leq m < n, \gcd(n,m) = 1\}$

$* \equiv$ multiplication modulo n

Denoted as $(Z_n^*, *_n)$

Required properties:

- Closure. Yes.
- Associativity. Yes.
- Identity. 1.
- Inverse. Yes.

Example: $Z_{15}^* = \{1,2,4,7,8,11,13,14\}$

$1^{-1} = 1,\ 2^{-1} = 8,\ 4^{-1} = 4,\ 7^{-1} = 13,\ 11^{-1} = 11,\ 14^{-1} = 14$

The Euler Phi Function

$$\phi(n) = \left|Z_n^*\right| = n\prod_{p|n}(1-1/p)$$

If n is a product of two primes p and q, then

$$\phi(n) = pq(1-1/p)(1-1/q) = (p-1)(q-1)$$

Note that by Fermat's Little Theorem:

$$a^{\phi(n)} = 1 \pmod{n} \quad \text{for} \quad a \in Z_n^*$$

Or for n = pq

$$a^{(p-1)(q-1)} = 1 \pmod{n} \quad \text{for} \quad a \in Z_{pq}^*$$

This will be very important in RSA!

Generators

Example of Z_{10}^*: {1, 3, 7, 9}

Generators

x	x^2	x^3	x^4
1	1	1	1
3	**9**	**7**	**1**
7	**9**	**3**	**1**
9	1	9	1

For $n = (2, 4, p^e, 2p^e)$, p an odd prime, Z_n is cyclic

Operations we will need

Multiplication: $a*b \pmod n$

- Can be done in $O(\log^2 n)$ bit operations, or better

Power: $a^k \pmod n$

- The power method $O(\log n)$ steps, $O(\log^3 n)$ bit ops

```
fun pow(a,k) =
  if (k = 0) then 1
  else if (k mod 2 = 1)
       then a * (pow(a,k/2))^2
       else (pow(a, k/2))^2
```

Inverse: $a^{-1} \pmod n$

- Euclids algorithm $O(\log n)$ steps, $O(\log^3 n)$ bit ops

Euclid's Algorithm

Euclid's Algorithm:

gcd(a,b) = gcd(b,a mod b)

gcd(a,0) = a

"Extended" Euclid's algorithm:

- Find **x** and **y** such that **ax + by = gcd(a,b)**
- Can be calculated as a side-effect of Euclid's algorithm.
- Note that **x** and **y** can be zero or negative.

This allows us to find $\underline{\mathbf{a^{-1} \bmod n}}$, for **a** $\in Z_n^*$

In particular return $\underline{\mathbf{x}}$ in $\underline{\mathbf{ax + ny = 1}}$.

Euclid's Algorithm

```
fun euclid(a,b) =
  if (b = 0) then a
  else euclid(b, a mod b)

                                   gcd     y
fun ext_euclid(a,b) =
  if (b = 0) then (a, 1, 0)
                                        x
  else
    let (d, x, y) = ext_euclid(b, a mod b)
    in (d, y, x - (a/b) y)
    end
```

The code is in the form of an inductive proof.

<u>Exercise</u>: prove the inductive step

Discrete Logarithms

If g is a generator of Z_n^*, then for all y there is a unique x (mod $\phi(n)$) such that

- $y = g^x \bmod n$

This is called the **discrete logarithm** of y and we use the notation

- $x = \log_g(y)$

In general finding the discrete logarithm is conjectured to be hard. It is as hard as factoring.

Polynomials with coefficients in $GF(p^n)$

We can make a finite field by using an irreducible polynomial $M(x)$ selected from $GF(p^n)[x]$.

For an order m polynomial and by abuse of notation we write: **$GF(GF(p^n)^m)$**, which has p^{nm} elements.

Used in **Reed-Solomon codes** and **Rijndael**.

- In Rijndael $p=2$, $n=8$, $m=4$, i.e. each coefficient is a byte, and each element is a 4 byte word (32 bits).

Note: all finite fields are isomorphic to $GF(p^n)$, so this is really just another representation of $GF(2^{32})$. This representation, however, has practical advantages.

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Private-Key Algorithms:

- Block ciphers and product ciphers
- Rijndael, DES
- Cryptanalysis

Public-Key Algorithms: Knapsack, RSA, El-Gamal, …

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Private Key Algorithms

Plaintext

↓

Key_1 → **Encryption** $E_k(M) = C$

↓ **Cyphertext**

Key_1 → **Decryption** $D_k(C) = M$

↓

Original Plaintext

What granularity of the message does E_k encrypt?

Private Key Algorithms

Block Ciphers: blocks of bits at a time

- DES (Data Encryption Standard)
  Banks, linux passwords (almost), TSL, kerberos, …
- Blowfish
- IDEA (used in PGP, TSL as option)
- Rijdael (AES) – **the current standard**

Stream Ciphers: one bit (or a few bits) at a time

- RC4 (TSL as option)
- PKZip
- Sober, Leviathan, Panama, …

Private Key: Block Ciphers

Encrypt one block at a time (e.g. 64 bits)

$c_i = f(k,m_i) \qquad m_i = f'(k,c_i)$

Keys and blocks are often about the same size.

Equal message blocks will encrypt to equal codeblocks

- Why is this a problem?

Various ways to avoid this:

- E.g. $c_i = f(k,c_{i-1} \text{ xor } m_i)$
  "**Cipher block chaining**" (CBC)

Why could this still be a problem?

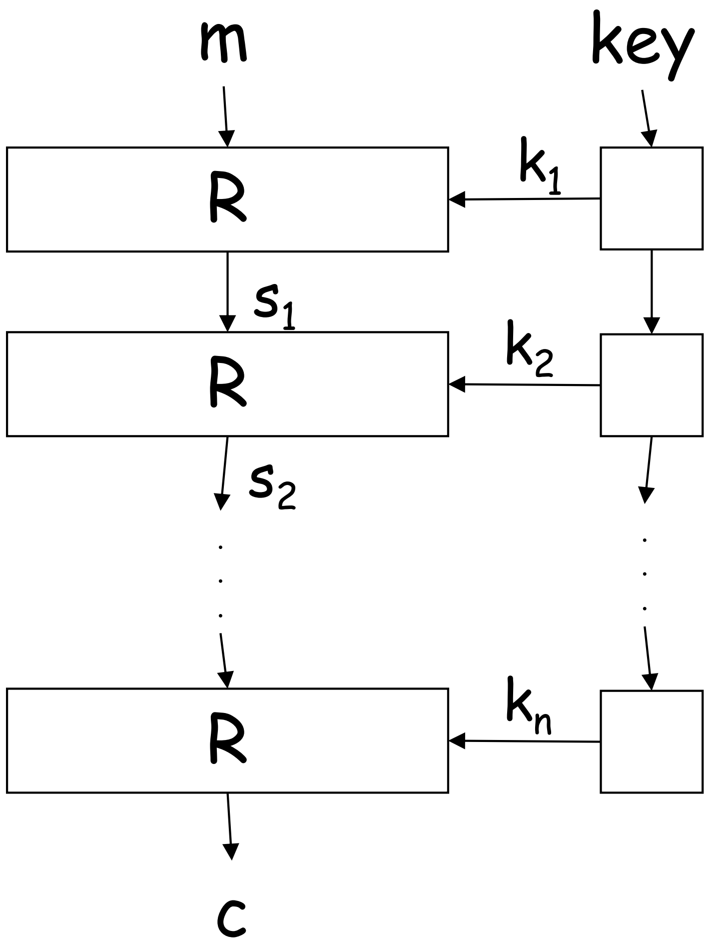
Solution: attach random block to the front of the message

Security of block ciphers

Ideal:

- k-bit -> k-bit key-dependent subsitution (i.e. "random permutation")
- If keys and blocks are k-bits, can be implemented with 2^{2k} entry table as a key!!!!! Completely impractical.

Iterated Block Ciphers



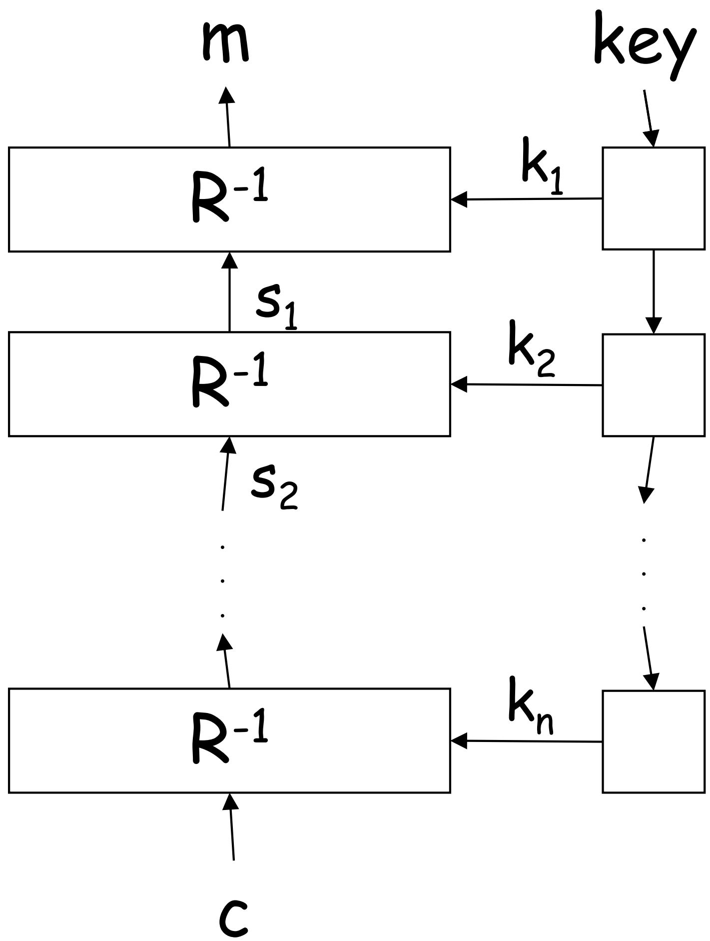
Consists of n **rounds**

R = the "round" function

s_i = state after round i

k_i = the i^{th} round key

Iterated Block Ciphers: Decryption

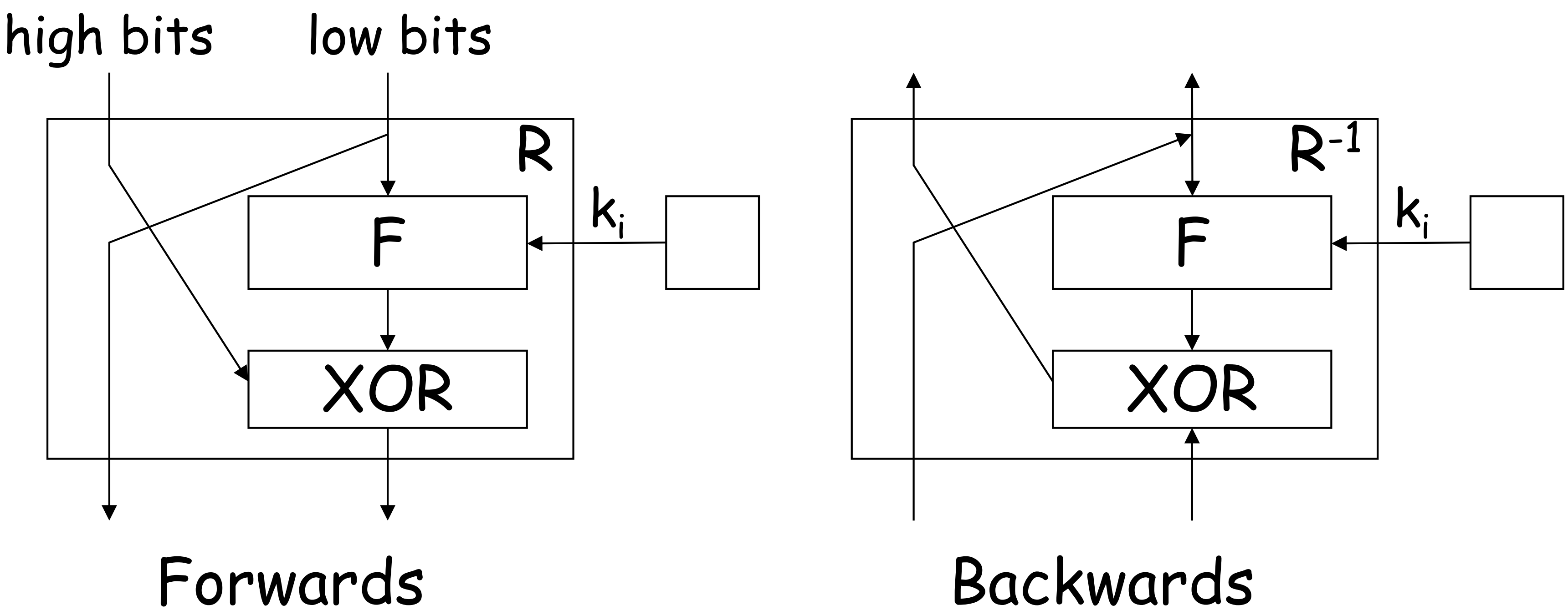


Run the rounds in reverse.

Requires that R has an inverse.

Feistel Networks

If function is not invertible rounds can still be made invertible. Requires 2 rounds to mix all bits.



Used by DES (the Data Encryption Standard)

Product Ciphers

Each round has two components:

- **Substitution** on smaller blocks
  Decorrelate input and output: “confusion”
- **Permutation** across the smaller blocks
  Mix the bits: “diffusion”

Substitution-Permutation Product Cipher

Avalanche Effect: 1 bit of input should affect all output bits, ideally evenly, and for all settings of other in bits

Rijndael (AES)

Selected by AES (Advanced Encryption Standard, part of NIST) as the new private-key encryption standard in 2002 over DES.

Based on an open "competition".

- Competition started Sept. 1997.
- Narrowed to 5 Sept. 1999
  - MARS by IBM, RC6 by RSA, Twofish by Counterplane, Serpent, and Rijndael
- Rijndael selected Oct. 2000.
- Official May 2002 (AES page on Rijndael)

Designed by Rijmen and Daemen (Dutch)

Goals of Rijndael

Resistance against known attacks:

- Differential cryptanalysis
- Linear cryptanalysis
- Truncated differentials
- Square attacks
- Interpolation attacks
- Weak and related keys

Speed + Memory efficiency across platforms

- 32-bit processors
- 8-bit processors (e.g smart cards)
- Dedicated hardware

Design simplicity and clearly stated security goals

High-level overview

An iterated block cipher with

- 10–14 rounds,
- 128-256 bit blocks, and
- 128-256 bit keys

Mathematically reasonably sophisticated

Blocks and Keys

The blocks and keys are organized as matrices of bytes. For the 128-bit case, it is a 4x4 matrix.

$$\begin{pmatrix} b_0 & b_4 & b_8 & b_{12} \\ b_1 & b_5 & b_9 & b_{13} \\ b_2 & b_6 & b_{10} & b_{14} \\ b_3 & b_7 & b_{11} & b_{15} \end{pmatrix} \qquad \begin{pmatrix} k_0 & k_4 & k_8 & k_{12} \\ k_1 & k_5 & k_9 & k_{13} \\ k_2 & k_6 & k_{10} & k_{14} \\ k_3 & k_7 & k_{11} & k_{15} \end{pmatrix}$$

Data block Key

b_0, b_1, …, b_{15} is the order of the bytes in the stream.

Galois Fields in Rijndael

Uses $GF(2^8)$ over bytes.

The irreducible polynomial is:

$M(x) = x^8 + x^4 + x^3 + x + 1$ or 100011011 or 0x11B

Also uses degree 3 polynomials with coefficients from $GF(2^8)$.

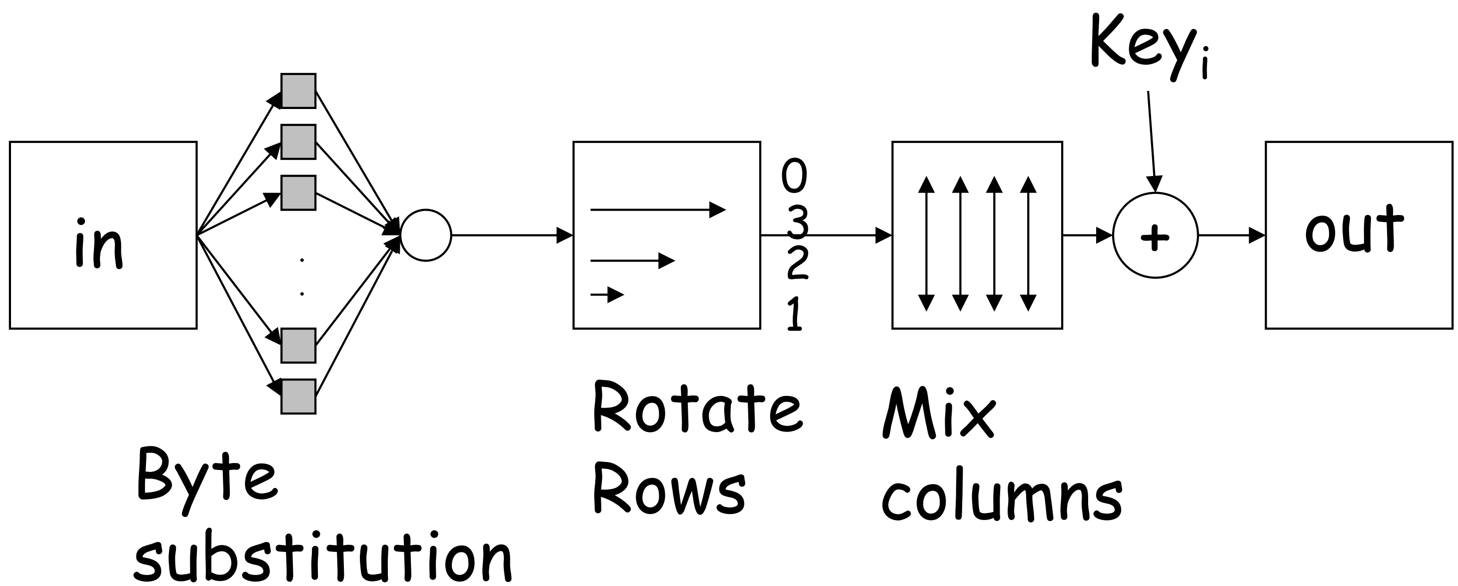
These are kept as 4 bytes (used for the columns)

The polynomial used as a modulus is:

$M(x) = 00000001x^4 + 00000001$ or $x^4 + 1$

Not irreducible, but we only need to find inverses of polynomials that are relatively prime to it.

Each round



The inverse runs the steps and rounds backwards.
Each step must be reversible!

Byte Substitution

Non linear: $y = b^{-1}$ (done over $GF(2^8)$)

Linear: $z = Ay + B$ (done over $GF(2)$, *i.e.*, binary)

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ & & & \vdots & & & & \end{pmatrix} \qquad B = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

To invert the substitution:

$y = A^{-1}(z - B)$ (the matrix A is nonsingular)

$b = y^{-1}$ (over $GF(2^8)$)

Mix Columns

For each column a in data block $\begin{matrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{matrix}$

compute $b(x) = (a_3x^3+a_2x^2+a_1x+a_0)(3x^3+x^2+x+2) \bmod x^4+1$

where coefficients are taken over $GF(2^8)$.

New column b is $\begin{matrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{matrix}$ where $b(x)=b_3x^3+b_2x^2+b_1x+b_0$

Implementation

Using $x^j \bmod (x^4 + 1) = x^{(j \bmod 4)}$

$(a_3x^3+a_2x^2+a_1x+a_0)(3x^3+x^2+x+2) \bmod x^4+1$

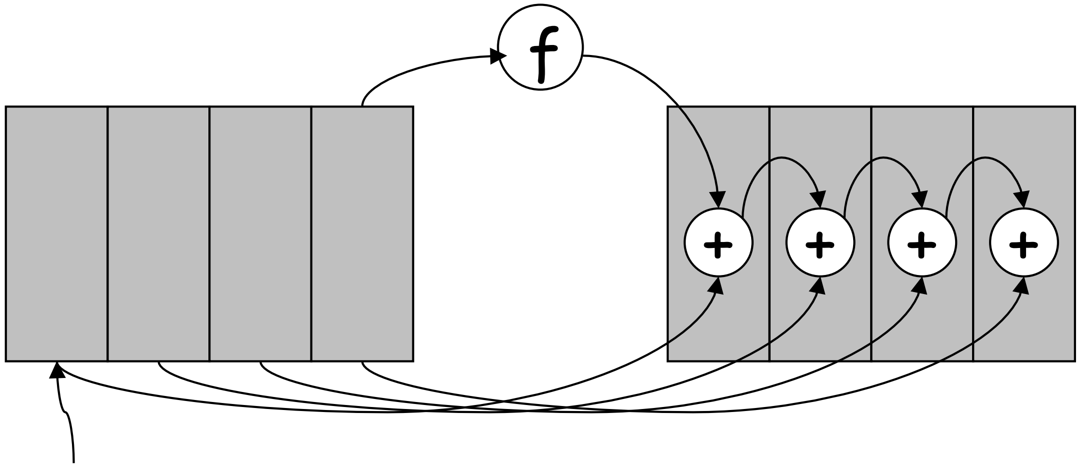
$$\begin{aligned} = &(2a_0+3a_1+a_2+a_3) + \\ &(a_0+2a_1+3a_2+a_3)x + \\ &(a_0+a_1+2a_2+3a_3)x^2 + \\ &(3a_0+a_1+a_2+2a_3)x^3 \end{aligned}$$

Therefore, $b = C \bullet a$

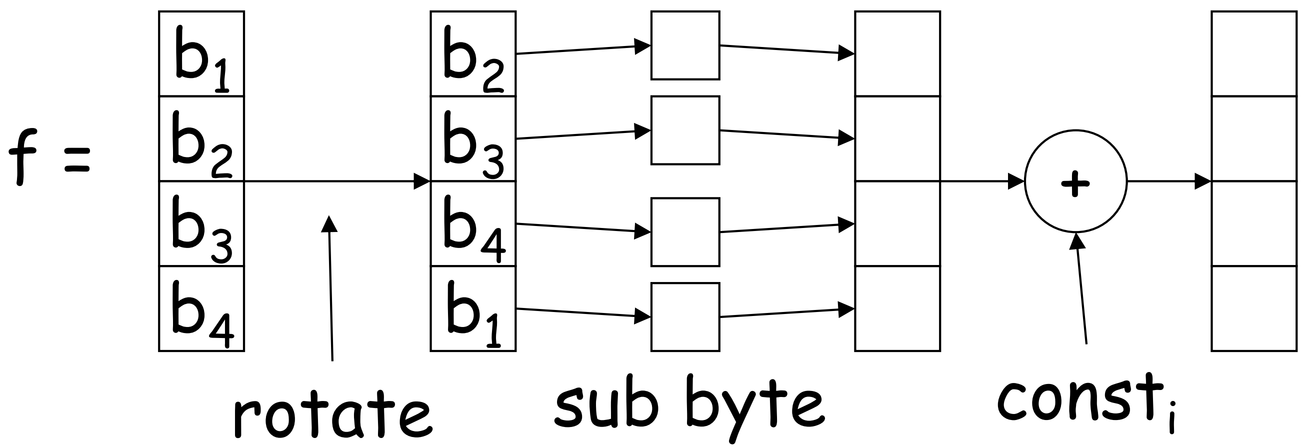
$$C = \begin{pmatrix} 2 & 3 & 1 & 1 \\ 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 3 \\ 3 & 1 & 1 & 2 \end{pmatrix}$$

M(x) is not irreducible, but the rows of C and M(x) are coprime, so the transform can be inverted.

Generating the round keys



Words corresponding to columns of the key



Performance

Performance: (64-bit AMD Athlon 2.2Ghz, 2005, Open SSL):

Algorithm	Bits/key	Mbits/sec
DES-cbc	56	399
Blowfish-cbc	128	703
Rijndael-cbc	128	917

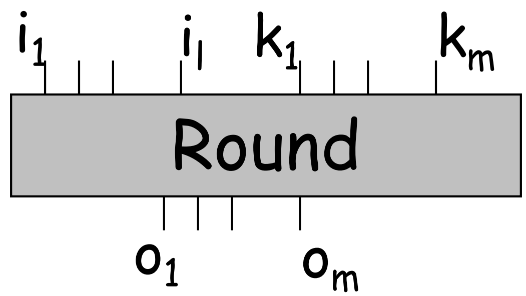
Intel X86 now has AES instructions (since 2008)
With instructions Intel-i7 gives 12Gbits/sec/core

X86 Instructions

AESENC	Perform one round of an AES encryption flow
AESENCLAST	Perform the last round of an AES encryption flow
AESDEC	Perform one round of an AES decryption flow
AESDECLAST	Perform the last round of an AES decryption flow
AESKEYGENASSIST	Assist in AES round key generation
AESIMC	Assist in AES Inverse Mix Columns
PCLMULQDQ	Carryless multiply (CLMUL)[3]

Linear Cryptanalysis

A known plaintext attack used to extract the key



Consider a linear equality involving i, o, and k

- e.g.: $k_1 + k_6 = i_2 + i_4 + i_5 + o_4$

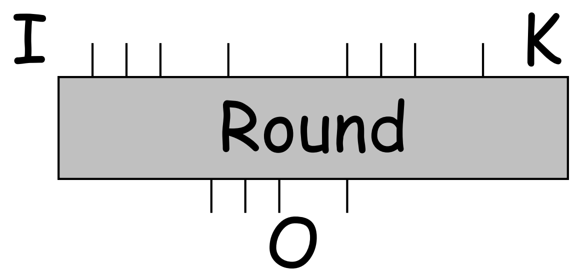
To be secure this should be true with $p = .5$ (probability over all inputs and keys)

If true with $p = 1$, then linear and easy to break

If true with $p = .5 + \varepsilon$ then you might be able to use this to help break the system

Differential Cryptanalysis

A chosen plaintext attack used to extract the key



Considers fixed "differences" between inputs, $\Delta_I = I_1 - I_2$, and sees how they propagate into differences in the outputs, $\Delta_O = O_1 - O_2$. "difference" is often exclusive OR

Assigns probabilities to different keys based on these differences. With enough and appropriate samples (I_1, I_2, O_1, O_2), the probability of a particular key will converge to 1.