1 Chernoff Bounds Redux

(a) When $p_i = 1/2$, $\mu = n/2$. Plugging in we have:

$$\mathbf{Pr}[X \geq (1+\delta)\mu] \leq (\frac{e^\delta}{(1+\delta)^{1+\delta}})^{n/2}$$

(b) When $X_i \sim U(0,1)$, $E[X_i] = 1/2$ and so $E[X] = n/2$ by linearity of expectation.

(c) We evaluate $E[e^{tX}]$ in the case when X is a sum of uniform r.v.'s and then plug this back into the proof of the Chernoff bound from class. We first calculate $E[e^{tX_i}]$:

$$\begin{aligned} E[e^{tX_i}] &= \int_0^1 e^{tx} dx \\ &= \frac{e^t}{t} \end{aligned}$$

Plugging in $t = \ln(1+\delta)$, we have:

$$E[e^{tX_i}] = \frac{\delta}{\ln(1+\delta)}$$

Recall that

$$E[e^{tX}] = \Pi_i E[e^{tX_i}] = (\frac{\delta}{\ln(1+\delta)})^n$$

Finishing off the Chernoff bound proof, we get

$$\mathbf{Pr}[X \geq (1+\delta)\mu] \leq (\frac{\delta^2}{\ln(1+\delta)^2(1+\delta)^{1+\delta}})^{n/2}$$

The bound is better than the bound for Bernoulli r.v.'s as a little calculation shows. The reason is because the variance of the uniform distribution is smaller than the variance of a Bernoulli RV.

2 Angle-preserving JL

As mentioned in the homework, the angles for skinny triangles can be distorted significantly. In order to prevent this we try to add extra points in such a way that if the distances between these points are approximately preserved, the *height* of the triangle cannot change that much. This then lets us argue that the $\sin$ of all angles are not significanlty distorted. Some extra work is needed to argue that the angles themeselves are not significantly distorted, as done in Avner Magen's paper [1]. The proof below is from the aforementioned paper.

[1] Dimensionality Reductions in l_2 that Preserve Volumes and Distance to Affine Spaces, Avner Magen

Consider a triple of points, $a, b, c \in P$ which define a triangle. In what follows, we only deal with acute angles, but the same argument will apply to an obtuse triangle by considering the acute angle induced by the obtuse angle and the line formed from one of its incident line segments.

For a set $S \subseteq \mathbb{R}^n$, let $\mathcal{L}(S)$ be the affine-hull of S, i.e. $\mathcal{L}(S) = \{\sum_{i=1}^{|S|} \lambda_i a_i | \sum \lambda_i = 1\}$. Let $P(x, S)$ be the *projection* of x onto $\mathcal{L}(S)$. The *affine distance*, or *height* of a vector $x \in \mathbb{R}^n$ to a set S is denoted $\textsc{ad}(x, S)$ and is just the distance of x to the affine-hull of S, i.e. $||x - P(x, S)||$.

It helps to only consider *contracting* embeddings, i.e. embeddings that do not increase the distance between two points in the original space w.h.p. Formally, we use the following version of JL:

$$\mathbf{Pr}[\frac{1}{1+\epsilon} \leq \frac{||f(v)||}{||v||} \leq 1] \geq 1 - \exp(t\epsilon^2) \tag{1}$$

where $t = O(\log n)$ and $\epsilon \leq 1/3$.

We first argue that right isoceles triangles are *stable*, i.e. a contracting embedding cannot mess up their height too much.

Lemma 1 *Let A, B and C be the vertices of a right isoceles triangle where the right angle is at A. Let ϕ be a contracting embedding (described above). Let H be the length of $|AB|$ (this is the original height). Then:*

$$\frac{1 - 3\epsilon^2}{1+\epsilon} \cdot H \leq \textsc{ad}(\phi(C), \{\phi(A), \phi(B)\}) \leq H$$

The $\textsc{ad}$ quantity is just the height of the triangle after applying ϕ. What this lemma is saying is that the height of the triangle is approximately preserved by the embedding. Let $\vec{b}$ be the vector $\phi(B) - \phi(A)$, and $\vec{c} = \phi(C) - \phi(A)$. Let $a = ||\vec{b} - \vec{c}||$ (this is the length of the side opposite $\phi(A)$). Let θ be the angle between $\vec{b}, \vec{c}$ (before applying phi, θ was the right angle). We have upper and lower bounds on each of the sidelengths based on Equation 1, and want to relate these to θ. If we can relate these to $\sin\theta$, we can recover the height of the mapped triangle.

The worst distortion occurs when $\sin\theta$ is minimized (i.e. it is far from 1). As $\sin\theta = \sqrt{1 - \cos^2(\theta)}$, we minimize the sin by maximizing $\cos\theta$. Given the side lengths some calculation using the law of cosines ($|\cos\theta| = \frac{|\|\vec{b}\|^2 + \|\vec{c}\|^2 - a^2|}{2\|\vec{b}\|\|\vec{c}\|}$) shows that the values of $a, ||\vec{b}||, ||\vec{c}||$ that maximize $|\cos\theta|$ are $||\vec{b}|| = ||\vec{c}|| = \frac{H}{1+\epsilon}$ and $a = \sqrt{2}H$ (a does not contract at all, and $\vec{b}$ and $\vec{c}$ contract fully).

Plugging the values of $a, \|\vec{b}\|, \|\vec{c}\|$, we have: $|\cos\theta| \leq |(2 - 2(1+\epsilon)^2|/2 = 2\epsilon + \epsilon^2$. Therefore:

$$\sin\theta = \sqrt{1 - \cos^2\theta} \geq \sqrt{1 - 4\epsilon^2 - 4\epsilon^3 - \epsilon^4} > 1 - 3\epsilon^2$$

Finishing off, we just need to relate $\sin\theta$ to H. As Figure 1 shows, the height is just $c\sin\theta$. Therefore, we have:

$$H \geq \textsc{ad}(\phi(C), \{\phi(A), \phi(B)\}) = c\sin\theta \geq \frac{1 - 3\epsilon^2}{1+\epsilon} \cdot H$$

The next step is to add 'supports' to a skinny bad triangle. Consider a triangle defined by a triple (a, b, c). For each pair, (x, y) and third point z, we add two additional points: $P(z, \{x, y\})$ and $\textsc{cor}(z, \{x, y\})$ which is a point on the line induced by $\bar{xy}$ whose distance from $P(z, \{x, y\})$ is $\textsc{ad}(z, \{x, y\})$. First we account for

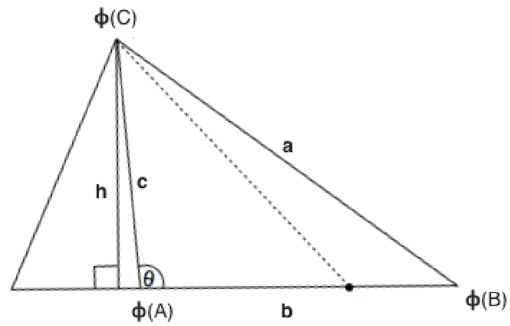


Figure 1: Note that $\sin \pi - \theta = \sin \theta$, and so the height h is given by $c \sin \theta$.

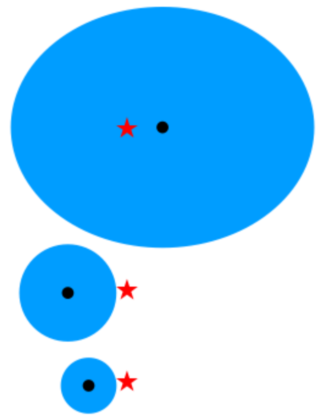
how many points this adds: there are $O(n^3)$ triples, and for each we add 6 points. Applying JL on this set of points increases the target dimension by a constant factor.

Next, we need to argue that our supported triangles will not have their angles significantly distorted. We can do this by comparing the $\sin$ of an angle α before and after the transformation. Let $\alpha = \measuredangle ABC$ and $\alpha' = \measuredangle \phi(A)\phi(B)\phi(C)$. We have that $\sin \alpha = \frac{\text{AD}(C,\{A,B\})}{\|C-B\|}$ and similarly $\sin \alpha' = \frac{\text{AD}(\phi(C),\{\phi(A),\phi(B)\})}{\|\phi(C)-\phi(B)\|}$ (using notation from Figure 1). The ratio is therefore

$$\frac{\sin \alpha'}{\sin \alpha} = \frac{\|C - B\|}{\|\phi(C) - \phi(B)\|} \cdot \frac{\text{AD}(\phi(C), \{\phi(A), \phi(B)\})}{\text{AD}(C, \{A, B\})}$$

Applying Equation 1 to the first fraction gives us that this quantity is upper bounded by $1 + \epsilon$. Applying Lemma 1 to the second fraction gives us our lower bound; putting both together we have

$$\frac{1 - 3\epsilon^2}{1 + \epsilon} \leq \frac{\sin \alpha'}{\sin \alpha} \leq 1 + \epsilon$$



3 Cover Tree Insertion

A simple example is shown above. The red point (the star) is the one being inserted.