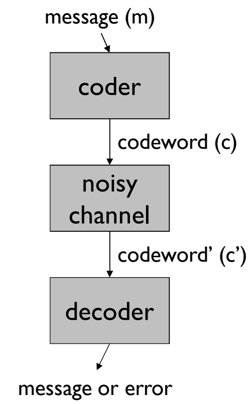
15-853:Algorithms in the Real World

Error Correcting Codes I
- Overview, Hamming Codes, Linear Codes

Error Correcting Codes II
- Reed-Solomon Codes
- Concatenated Codes

Error Correcting Codes III (LDPC/Expander Codes)
Error Correcting Codes IV (Decoding RS, Number thy)

Block Codes



Each message and codeword is of fixed size

Σ = codeword alphabet

k = $|m|$ **n** = $|c|$ **q** = $|\Sigma|$

C $\subseteq \Sigma^n$ (codewords)

Δ(x,y) = number of positions s.t. $x_i \neq y_i$

d = $\min\{\Delta(x,y) : x,y \in C, x \neq y\}$

Code described as: $\mathbf{(n,k,d)_q}$

Linear Codes

If Σ is a field, then Σ^n is a vector space

Definition: C is a linear code if it is a linear subspace of Σ^n of dimension k.

This means that there is a set of k independent vectors $v_i \in \Sigma^n$ $(1 \leq i \leq k)$ that span the subspace.

i.e. every codeword can be written as:

$$c = a_1 v_1 + a_2 v_2 + \ldots + a_k v_k \quad \text{where } a_i \in \Sigma$$

"Linear": the sum of two codewords is a codeword.

Minimum distance = weight of least-weight codeword

Generator and Parity Check Matrices

Generator Matrix:

A k x n matrix **G** such that: $C = \{ xG \mid x \in \Sigma^k \}$

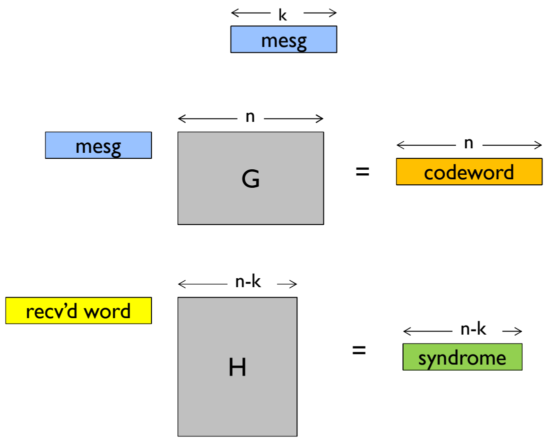
Made from stacking the spanning vectors

Parity Check Matrix:

An (n – k) x n matrix **H** such that: $C = \{y \in \Sigma^n \mid Hy^T = 0\}$

(Codewords are the null space of H.)

These **always exist for linear codes**



if syndrome = 0, received word = codeword

else have to use syndrome to get back codeword

Relationship of G and H

For linear codes, if G is in standard form $[I_k\ A]$
then $H = [-A^T\ I_{n-k}]$

Example of (7,4,3) Hamming code:

transpose

$$G = \left[\begin{array}{cccc|ccc} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right] \qquad H = \left[\begin{array}{cccc|ccc} 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

Two Codes

Hamming codes are binary $(2^r-1-1, 2^r-1-r, 3)$ codes.
Basically (n, n – log n, 3)
Hadamard codes are binary $(2^r-1, r, 2^{r-1})$.
Basically (n, log n, n/2)

The first has great rate, small distance.
The second has poor rate, great distance.
Can we get Ω(n) rate, Ω(n) distance?

Yes. One way is to use a random linear code.
Let's see some direct, intuitive ways.

Reed-Solomon Codes

Irving S. Reed and Gustave Solomon

PDF-417

QR code

Aztec code

DataMatrix code

All 2-dimensional Reed-Solomon bar codes

images: wikipedia

Reed-Solomon Codes in the Real World

$(204,188,17)_{256}$: ITU J.83(A)[2]
$(128,122,7)_{256}$: ITU J.83(B)
$(255,223,33)_{256}$: Common in Practice

- Note that they are all byte based (i.e., symbols are from $GF(2^8)$).

Decoding rate on 1.8GHz Pentium 4:

- (255,251) = 89Mbps
- (255,223) = 18Mbps

Dozens of companies sell hardware cores that operate 10x faster (or more)

- (204,188) = 320Mbps (Altera decoder)

Applications of Reed-Solomon Codes

- **Storage**: CDs, DVDs, “hard drives”,
- **Wireless**: Cell phones, wireless links
- **Sateline and Space**: TV, Mars rover, Voyager,
- **Digital Television**: DVD, MPEG2 layover
- **High Speed Modems**: ADSL, DSL, ..

Good at handling burst errors.

Other codes are better for random errors.

- e.g., Gallager codes, Turbo codes

Viewing Messages as Polynomials

A (n, k, n-k+1) code:

Consider the polynomial of degree k-1

$p(x) = a_{k-1} x^{k-1} + \cdots + a_1 x + a_0$

Message: $(a_{k-1}, \ldots, a_1, a_0)$

Codeword: $(p(1), p(2), \ldots, p(n))$

To keep the p(i) fixed size, we use $a_i \in GF(q^r)$

Finite field with q^r elements, q is a prime, r integer

To make the i distinct, $n \le q^r$

For simplicity, imagine that $n = q^r$. So we have a $(n, k, n-k+1)_n$ code.

Alphabet size increasing with the codeword length. A little awkward. (But we can still do things.)

Which field to use

The general ideas will not depend on the field.

Assume we're working in $\mathbb{F}_q$ where q is a prime.

Just has to be large enough to evaluate at n points.

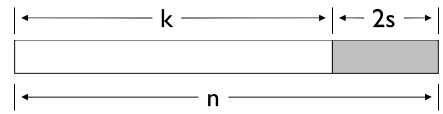
In practice, use $q = 2^r$, e.g., $\mathbb{F}_{2^8} = \mathbb{F}_{256}$.

Recall: this is **not** the same as the vector space $(\mathbb{F}_2)^8$.

Always think about $\mathbb{F}_{2^8}$ in terms of polynomials.

Polynomial-Based Code

A (n, k, 2s +1) code:



Can **detect** 2s errors

Can **correct** s errors

Generally can correct α erasures and β errors if $\alpha + 2\beta \leq 2s$

Polynomials and their degrees

Fundamental theorem of Algebra:

Any non-zero polynomial of degree k has at most k roots (over any field).

Corollary 1:

If two degree-k polynomials P, Q agree on k+1 locations (i.e., if $P(x_i) = Q(x_i)$ for $x_0, x_1, \ldots, x_k$), then P = Q.

Corollary 2:

Given any k+1 points (x_i, y_i), there is at most one degree-k polynomial that has $P(x_i) = y_i$ for all these i.

Polynomials and their degrees

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Theorem:

Given any k+1 points (x_i, y_i), there is exactly one degree-k polynomial that has $P(x_i) = y_i$ for all these i.

Proof: e.g., use Lagrange interpolation.

Why is the distance n-k+1?

Direct Proof

1. RS is a linear code: indeed, if we add two codewords corresponding to P(x) and Q(x), we get a codeword corresponding to the polynomial P(x) + Q(x).

2. So look at the least weight codeword. It is the evaluation of a polynomial of degree k-1 at some n points. So it can be zero on only k-1 points. Hence non-zero on (n-(k-1)) points.

3. This means distance at least d = n-k+1

Correcting Errors

Correcting s errors:

1. Find k+s symbols that agree on a degree (k-1) poly p(x). These must exist since originally k + 2s symbols agreed and only s are in error
2. There are no k+s symbols that agree on the wrong degree (k-1) polynomial p'(x)
   - Any subset of k symbols will define p'(x)
   - Since at most s out of the k+s symbols are in error, p'(x) = p(x)

Correct s errors, so distance ≥ 2s+1 = n-k+1

This suggests a brute-force approach, very inefficient.
Better algos exist (talk on the board).

The Berlekamp-Welch Decoder

An efficient way to decode Reed-Solomon codes.

We may do this in coding lecture #4 this time.

RS and "burst" errors

Let's compare to Hamming Codes (which are also optimal).

	code bits	check bits
RS $\mathbf{(255, 253, 3)_{256}}$	2040	16
Hamming $\mathbf{(2^{11}\text{-}1, 2^{11}\text{-}11\text{-}1, 3)_2}$	2047	11

They can both correct 1 error, but not 2 random errors.
- The Hamming code does this with fewer check bits

However, RS can fix 8 contiguous bit errors in one byte
- Much better than lower bound for 8 arbitrary errors

$$\log\left(1+\binom{n}{1}+\cdots+\binom{n}{8}\right) > 8\log(n-7) \approx 88 \text{ check bits}$$

One way to achieve linear rate and linear distance

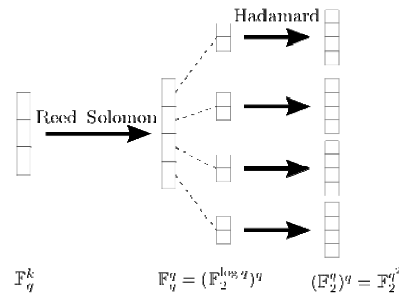
CONCATENATED CODES

Concatenated Codes

David Forney

Take a RS code $(n,k,n-k+1)_{q=n}$ code.

Can encode each alphabet symbol using another code.



Wikipedia

Concatenated Codes

Take a RS code $(n,k,n-k+1)_{q=n}$ code.

Can encode each alphabet symbol of k' = log q = log n bits using another code.

E.g., use $((k' + \log k'), k', 3)_2$-Hamming code. Now we can correct one error per alphabet symbol with little rate loss. (Good for sparse periodic errors.)

Or $(2^{k'}, k', 2^{k'-1})_2$ Hadamard code. (Say k = n/2.) Then get $(n^2, (n/2) \log n, n^2/4)_2$ code. Much better than plain Hadamard code in rate, distance worse only by factor of 2.

Concatenated Codes

Take any $(N,K,D)_{q^k}$ code.

Can encode each alphabet symbol of k bits using another $(n,k,d)_q$ code.

The concatenated code is a $(Nn, Kk, Dd)_q$ code

Concatenated Codes

Take a RS code $(n,k,n-k+1)_{q=n}$ code.

Can encode each alphabet symbol of $k' = \log q = \log n$ bits using another code.

Or, since k' is O(log n), could choose a code that requires exhaustive search but is good.

Random linear codes give $((1+f(\delta))k', k', \delta k'))_2$ codes.

Composing with RS (with k = n/2), we get

$((1+f(\delta))n \log n, (n/2) \log n, \delta(n/2)\log n)_2$

Gives constant rate and constant distance and binary alphabet! And poly-time encoding and decoding.