Variations on Nested Dissection

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Solving a Linear System

Problem Instance:

$$A \cdot x = b$$

Known $n \times n$ matrix

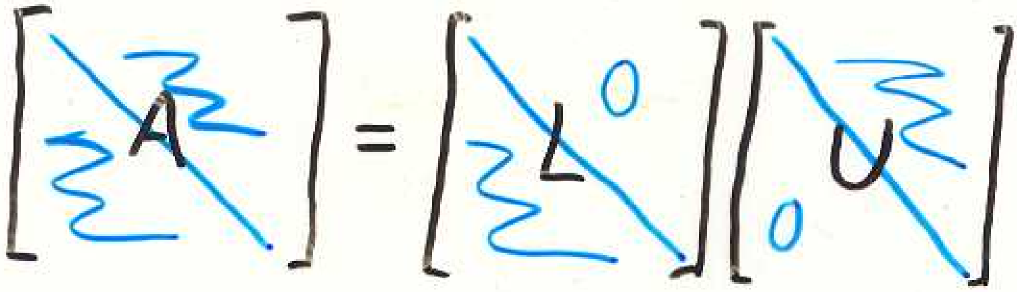
unknown $n \times 1$ vector

Known $n \times 1$ vector

Goal: Find x that satisfies equation.

Solving a Linear System

via LU decomposition



$$Ax = L\underbrace{Ux}_{y} = b$$

Solve $Ly = b$ for y,

(back substitution)

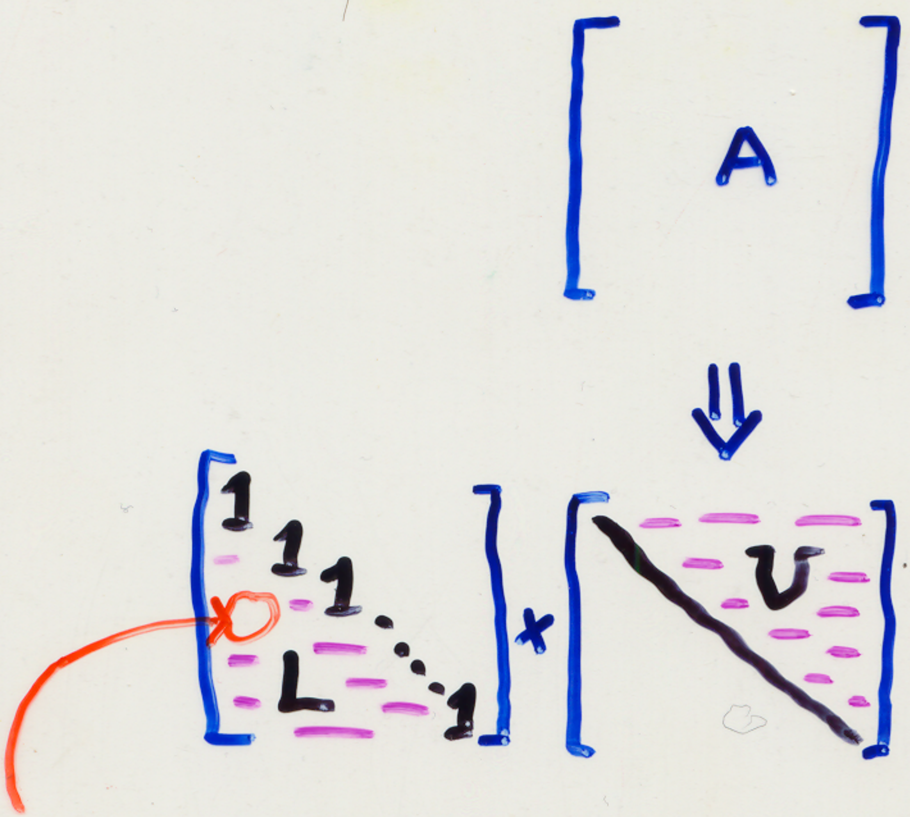
then solve $Ux = y$ for x.

Gaussian Elimination

$$
\begin{bmatrix}
1 & 3 & & & & \\
 & 1 & & & & \\
 & & 1 & 2 & -1 & 3 \\
 & & 2 & 2 & & \\
 & & -1 & & & 4 \\
 & & 3 & & 4 & 1
\end{bmatrix}
\begin{array}{l}
\\
\\
\\
-2\cdot\text{row } 3 \rightarrow \\
+1\cdot\text{row } 3 \rightarrow \\
-3\cdot\text{row } 3 \rightarrow
\end{array}
\begin{bmatrix}
1 & 3 & & & & \\
 & 1 & & & & \\
 & & 1 & 2 & -1 & 3 \\
 & & & -2 & 2 & -6 \\
 & & & 2 & -1 & 7 \\
 & & & -6 & 7 & -8
\end{bmatrix}
$$

pivot

LU Decomposition via Gaussian Elimination



L_{ij} is multiple of j'th pivot used to eliminate i'th entry in column j.

$$
\begin{bmatrix}
1 & 3 & & & & \\
 & 1 & & & & \\
 & & 1 & 2 & -1 & 3 \\
 & & 2 & 2 & & \\
 & & -1 & & & 4 \\
 & & 3 & & 4 & 1
\end{bmatrix}
$$

$$
\text{pivot} \quad
\begin{bmatrix}
1 & 3 & & & & \\
 & 1 & & & & \\
 & & 1 & 2 & -1 & 3 \\
 & & \cancel{2} & \cancel{2}\,{-2} & 2 & -6 \\
 & & \cancel{-1} & 2 & -1 & \cancel{4}\,7 \\
 & & \cancel{3} & -6 & \cancel{4}\,7 & \cancel{1}\,{-8}
\end{bmatrix}
\quad \begin{array}{l} \text{"fill"} \\ \text{(in green)} \end{array}
$$

$$\begin{bmatrix} 1 & 3 & & & & \\ & 1 & & & & \\ & & 1 & 2 & -1 & 3 \\ & & 2 & 2 & & \\ & & -1 & & & 4 \\ & & 3 & & 4 & 1 \end{bmatrix}$$

$\Downarrow$ reorder rows and columns

$$\begin{bmatrix} 1 & 3 & & & & \\ & 1 & & & & \\ & & 2 & 2 & & \\ & & 2 & 1 & -1 & 3 \\ & & & -1 & & 4 \\ & & & 3 & 4 & 1 \end{bmatrix}$$

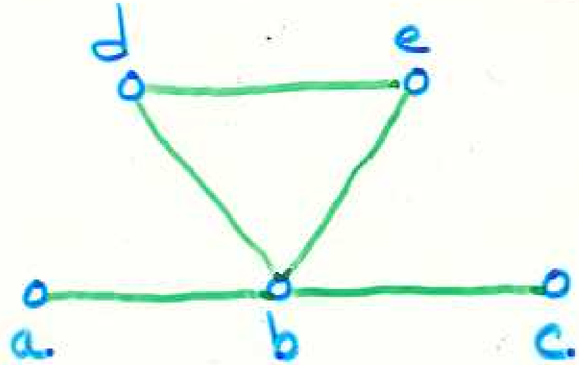
$$\begin{bmatrix} 1 & 3 & & & & \\ & 1 & & & & \\ & & 1 & 2 & -1 & 3 \\ & & 2 & 2 & & \\ & & -1 & & & 4 \\ & & 3 & & 4 & 1 \end{bmatrix}$$

$\Downarrow$ reorder rows and columns

$$\begin{bmatrix} 1 & 3 & & & & \\ & 1 & & & & \\ & & 2 & 2 & & \\ & & \cancel{2} & \cancel{1}\,{-1} & -1 & 3 \\ & & & -1 & & 4 \\ & & & 3 & 4 & 1 \end{bmatrix}$$

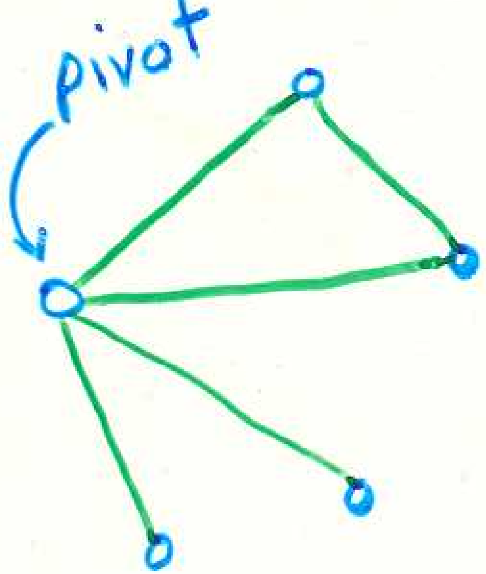
Graph Model

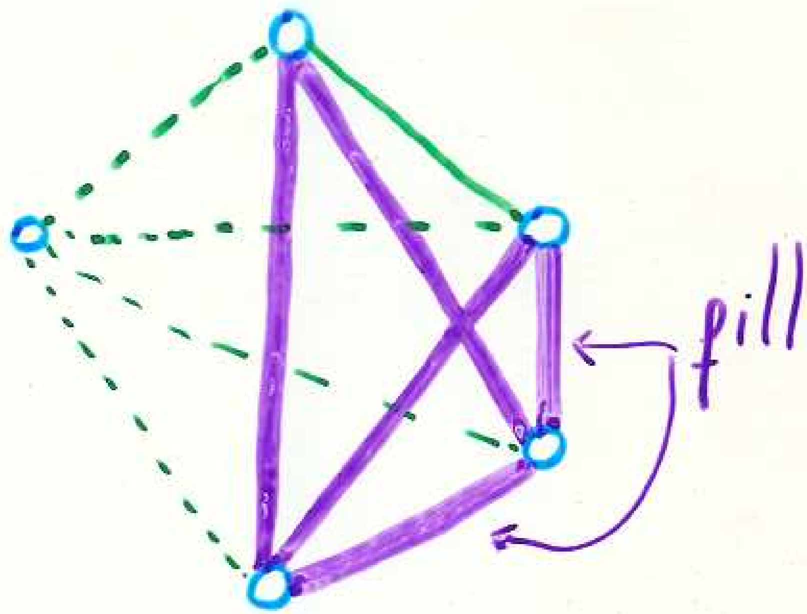
$$
\begin{array}{c|ccccc}
 & a & b & c & d & e \\
\hline
a & \times & \times & & & \\
b & \times & \times & \times & \times & \times \\
c & & \times & \times & & \\
d & & \times & & \times & \times \\
e & & \times & & \times & \times
\end{array}
$$



Symmetric matrix $\Longleftrightarrow$ undirected graph

Graph View of Pivoting





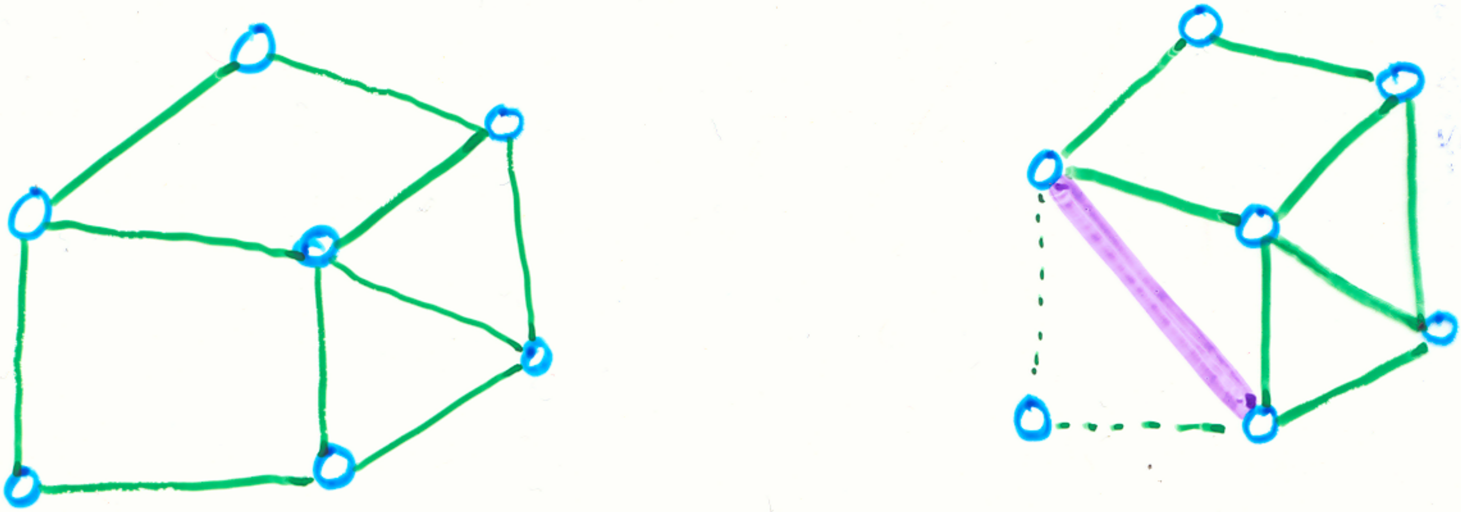
Disclaimer

$\underbrace{\text{Symmetric}}_{\substack{\text{undirected} \\ \text{graph}}}$, $\underbrace{\text{positive definite}}_{\substack{\text{pivots are nonzero} \\ \text{independent of order}}}$ matrices

Min-degree Heuristic

Eliminate vertex with smallest degree

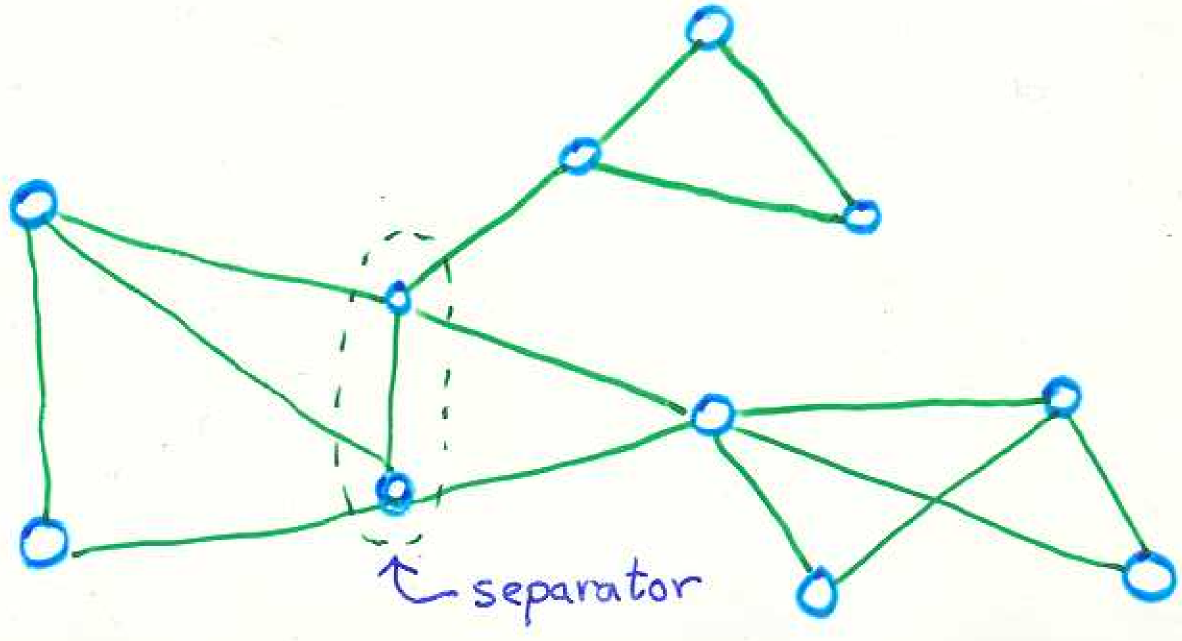
(Mankowitz, 57
Tinny & Walker, 67)



- simple, greedy algorithm
- works reasonably well in practice (doesn't create too much fill)

Separators

Idea: Eliminating separator last prevents fill between components



Nested Dissection

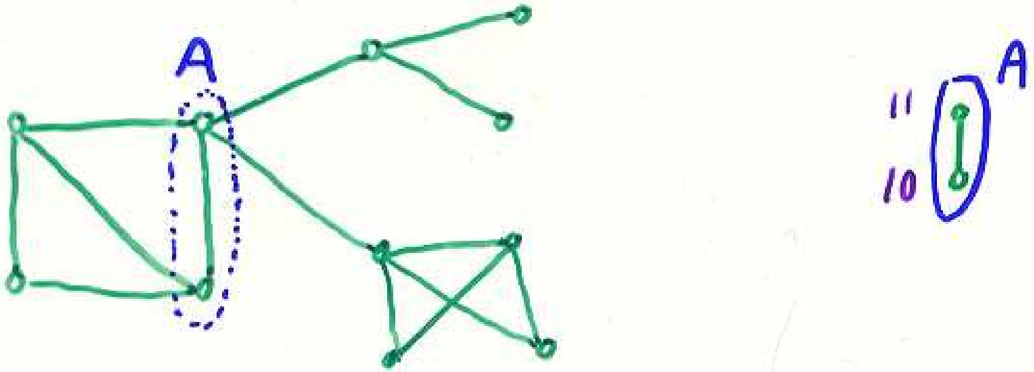
(George, 73; Lipton, Rose, Tarjan, 79)

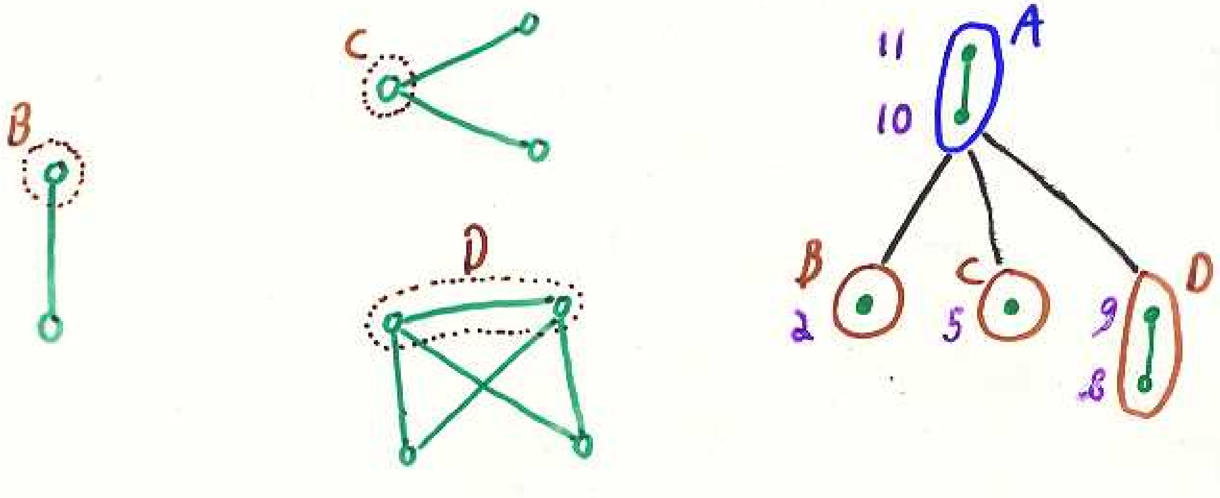
1. Find small node separator

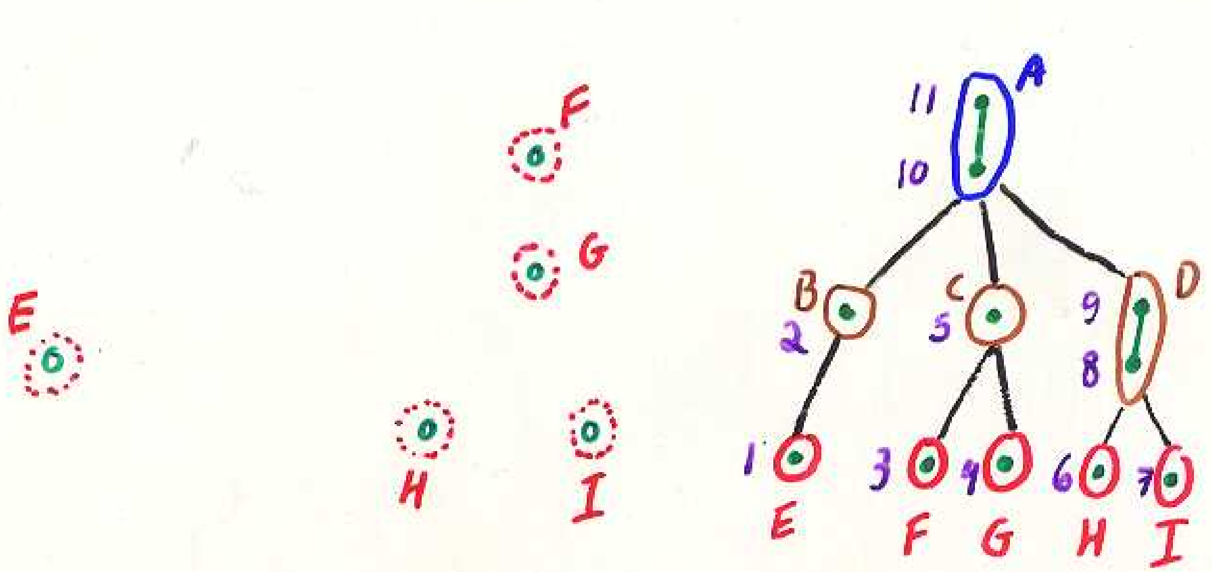
2. Place separator vertices last in the order

3. Recurse on connected components

Nested Dissection







Analysis of Nested Dissection

Lipton, Rose, Tarjan, 79.

→ $\sqrt{n}$ separator theorem

$\Downarrow$

$O(n.\log n)$ fill

$O(n^{3/2})$ work

Gilbert, Tarjan, 87

→ $O(n.\log n)$ fill, $O(n^{3/2})$ work

on planar graphs, graphs with bounded genus, and bounded degree & $O(\sqrt{n})$ separators

Analysis of Nested Dissection

Agrawal et al, 93

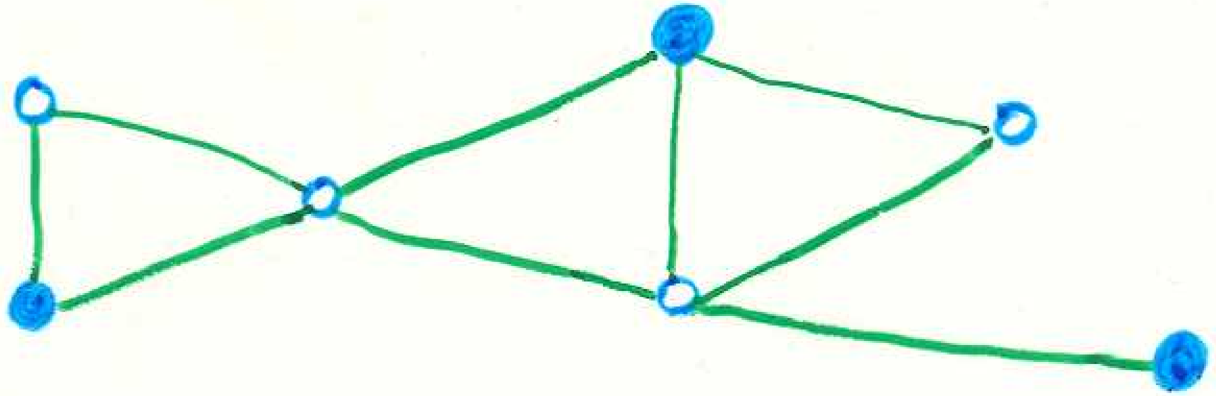
→ $O(\sqrt{d}\,\log^4 n) \times$ opt. non-zeros

→ $O(\log^2 n) \times$ opt. height

d = max. degree

Parallel Elimination

Observation: an independent set can be eliminated in parallel



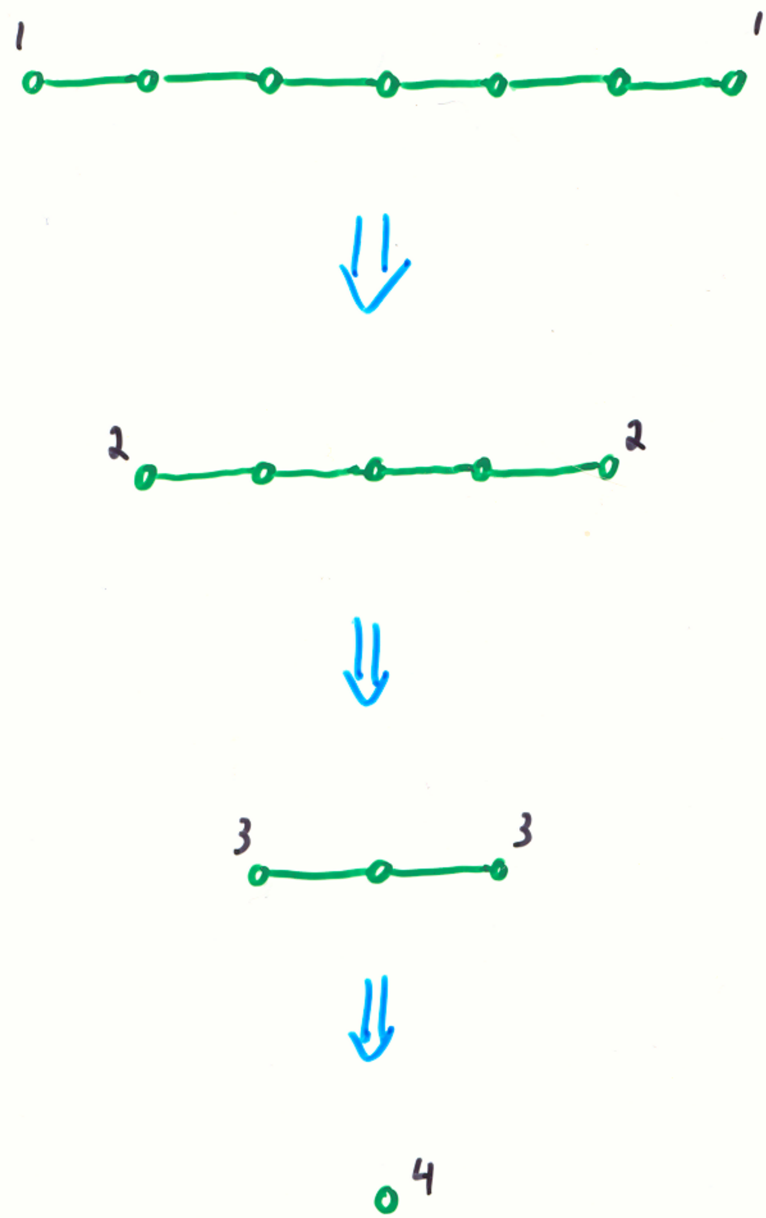
E.g., can eliminate one vertex from each component in nested dissection in parallel

Height

Def.: The height of an order is the number of parallel steps in the order.

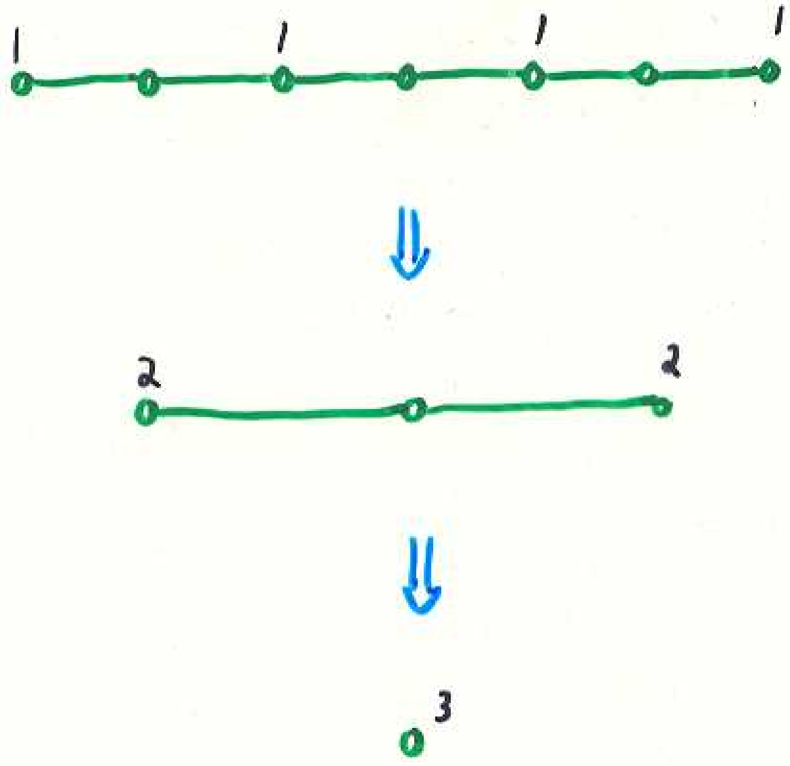
Fill versus Height

G



fill $= 0$ height $= \left\lfloor \frac{n}{2} \right\rfloor + 1$

Fill versus Height



$$\text{fill} = n - 2\lfloor \log n \rfloor - 1$$

$$\text{height} = \lfloor \log n \rfloor + 1$$

(for $n = 2^k - 1$)

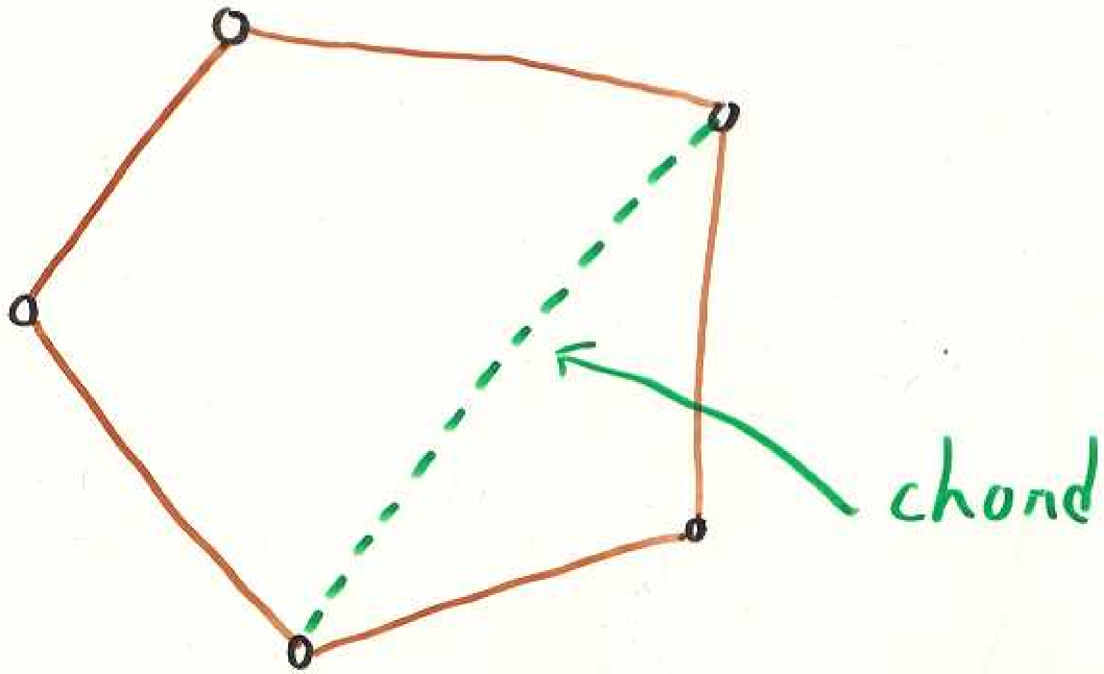
Outline

1. Parallel elimination orders for chordal graphs

2. Variation on nested dissection

Chordal Graphs

Definition: A graph G is chordal if and only if every cycle of length ≥ 4 has a chord



Chordal Graphs

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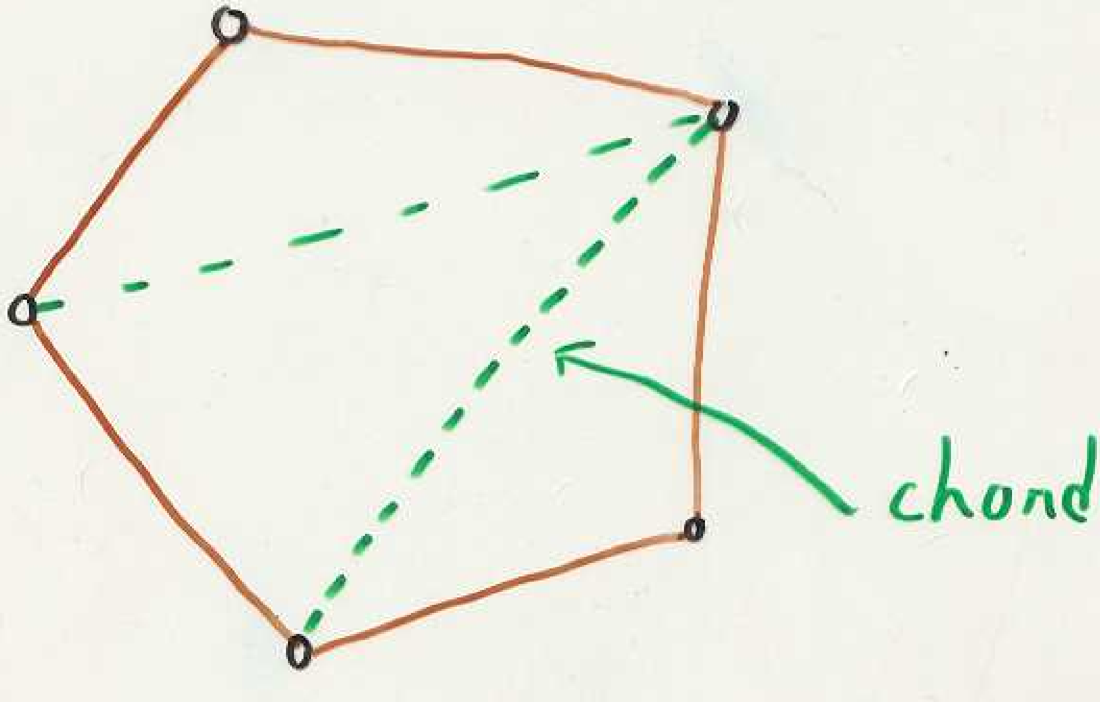
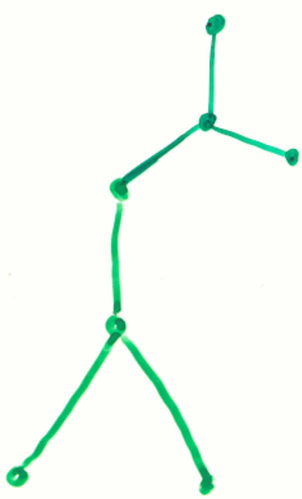


Chordal Graphs

Def. 2: G is chordal if and only if its nodes are connected subgraphs of a "skeleton" tree, and there is an edge between any two subgraphs that overlap.



Chordal Graphs

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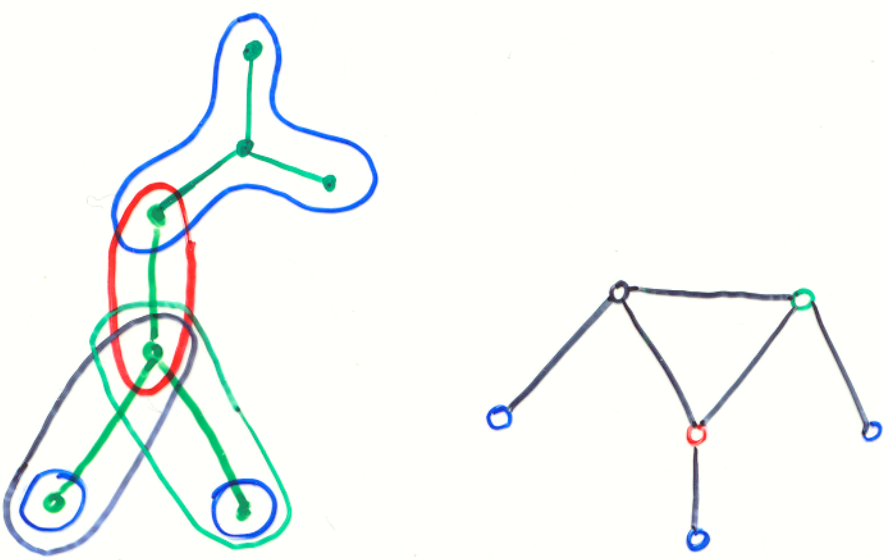


"INTERSECTION GRAPHS OF SUBTREES OF A TREE"

Chordal Graphs

Def. 3: G is chordal if and only if it has a zero-fill elimination order.

(also called a perfect elimination)

Our Results (Interval Graphs)

1. $\frac{1}{2}$-$\frac{1}{2}$ nested dissection yields

$\theta(m.\sqrt{\log n})$ fill

2. $\frac{1}{3}$-$\frac{2}{3}$ nested dissection yields

$\theta(m)$ fill

both with $\theta(\log n)$. opt height

n = # vertices

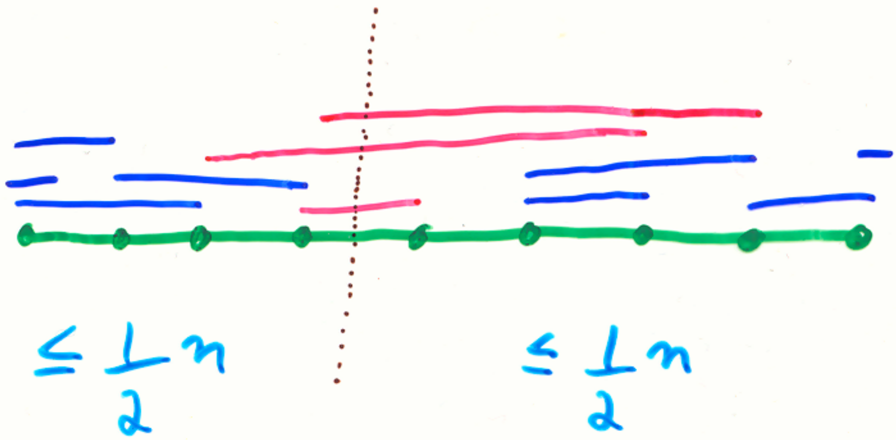
m = # edges

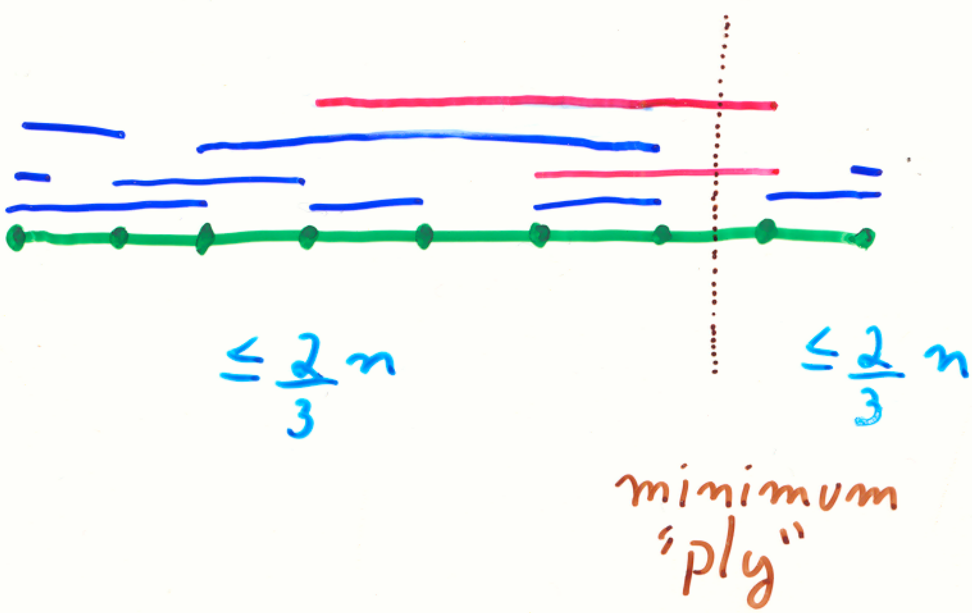
Our Results (Chordal Graphs)

$\frac{1}{3}$-$\frac{2}{3}$ nested dissection can create more than linear fill.

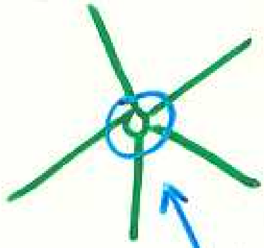
We find an $O(\log^2 n)$-opt height order with linear fill.

$\frac{1}{2}$-$\frac{1}{2}$ versus $\frac{1}{3}$-$\frac{2}{3}$ Separators





Bad Example for Chordal Graphs



only choice for

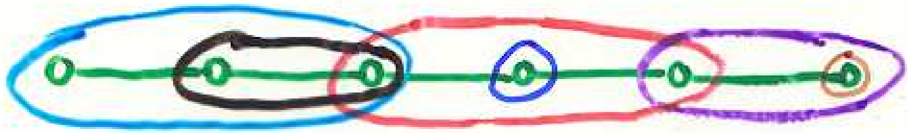
$\frac{1}{3}$-$\frac{2}{3}$ separator

Can force same behavior as in

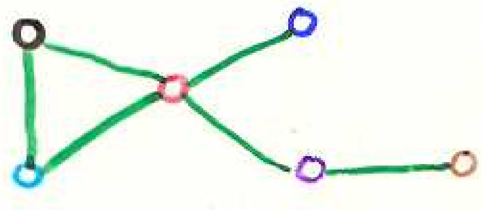
$\frac{1}{2}$-$\frac{1}{2}$ separators for interval graphs

Special Case: Interval Graphs

Skeleton tree is a chain

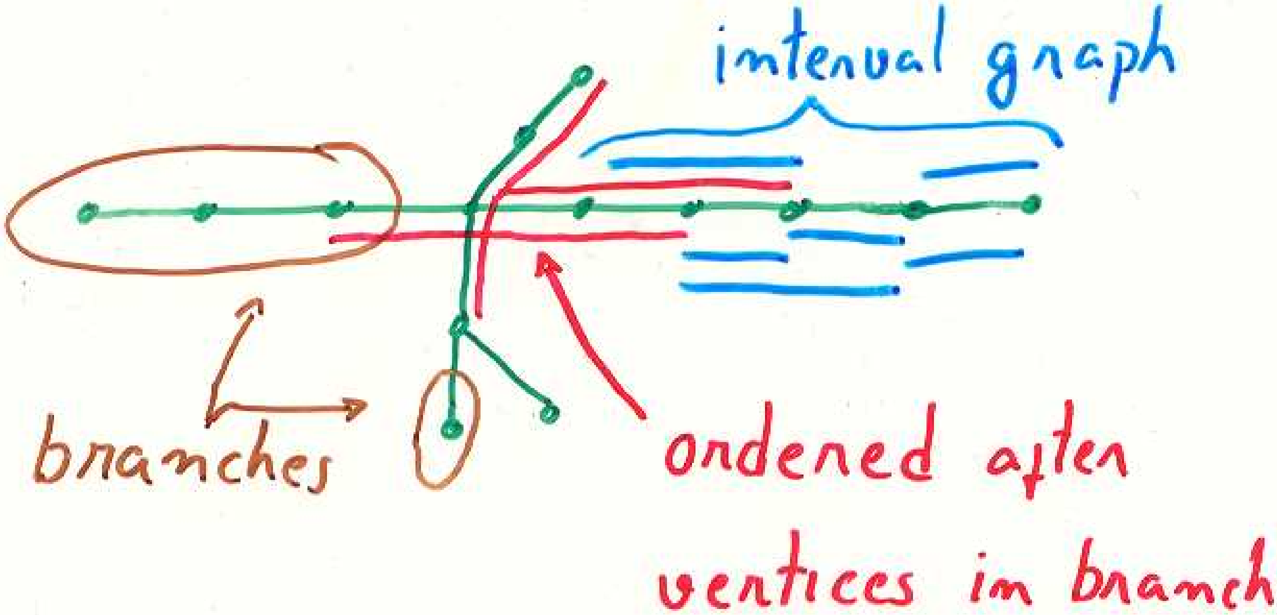


$\Downarrow$



Usually depict only intervals:

Chordal Graph Algorithm



Tree contraction: $O(\log n)$ "parallel pruning" of branches

Sentinels

Find sentinels right-to-left.

(whenever ply from external intervals doubles)

Order sentinels right-to-left

vertex in >1 sentinel goes in leftmost

Why are Chordal Graphs Interesting?

1. Rich structure

2. Every order for an arbitrary graph specifies a chordal completion of the graph.

Post-processing Idea

Apply our algorithm for parallelizing an order to the chordal completion produced by a low-fill but sequential order. Then use the new order for the original graph.

Experiments

Used three types of ordering algorithms:

1- Min-degree

2- Nested dissection

3- Hybrid.

Hybrid Algorithm (BEND, Hendrickson & Rothberg, 96)

1- Perform a few levels of Nested dissection until components are small.

2- Eliminate components with min-degree.

3- Combine all separators, order with min-degree.

Matrices

Matrix	vertices	edges
shuttle-eddy	10429	46585
sf10	7294	44922
bcsstk35	30237	709963
bcsstk37	25503	557737
bcsstk36	23052	560044
struct3	53570	560062
sf5	30169	190377
nasasrb	54870	1311227
cfd1	70656	878854
cfd2	123440	1482229

Fill

Matrix	vertices	edges	AMD ($\times 10^3$)	Post fill	BEND ($\times 10^3$)	Post fill	Metis ($\times 10^3$)	Post fill
shuttle-eddy	10429	46585	327	1.123	330	1.062	368	0.998
sf10	7294	44922	676	1.030	531	1.010	570	1.000
bcsstk35	30237	709963	2732	1.038	2776	1.011	3132	0.992
bcsstk37	25503	557737	2799	1.033	2693	1.017	3146	0.987
bcsstk36	23052	560044	2733	1.012	2555	1.022	3037	0.994
struct3	53570	560062	5093	1.013	4452	1.008	4574	0.992
sf5	30169	190377	5244	1.020	3873	1.083	3997	1.000
nasasrb	54870	1311227	11954	1.168	9765	1.051	10582	0.996
cfd1	70656	878854	37734	1.105	22386	1.011	22845	0.999
cfd2	123440	1482229	75008	1.014	39026	1.034	38937	1.000

Approximate Minimum Degree

Hybrid

Nested Dissection

Height

Matrix	vertices	edges	AMD	Post height	BEND	Post height	Metis	Post height
shuttle-eddy	10429	46585	851	0.677	659	0.730	335	0.988
sf10	7294	44922	793	0.974	703	0.983	557	1.000
bcsstk35	30237	709963	1262	0.948	1084	0.989	970	0.939
bcsstk37	25503	557737	1333	1.065	1162	1.022	1232	1.000
bcsstk36	23052	560044	1540	0.932	1255	1.042	1166	0.990
struct3	53570	560062	1542	0.982	1440	1.002	1093	0.992
sf5	30169	190377	2341	1.000	1830	1.033	1523	0.998
nasasrb	54870	1311227	4829	0.549	3360	0.725	1662	1.112
cfd1	70656	878854	7921	0.878	3438	1.003	3228	1.000
cfd2	123440	1482229	9494	1.000	5538	0.967	4068	1.000

Work

Matrix	vertices	edges	AMD ($\times 10^6$)	Post work	BEND ($\times 10^6$)	Post work	Metis ($\times 10^6$)	Post work
shuttle-eddy	10429	46585	17	1.514	18	1.237	23	0.998
sf10	7294	44922	137	1.090	71	1.025	80	1.002
bcsstk35	30237	709963	383	1.145	394	1.026	509	0.990
bcsstk37	25503	557737	532	1.131	477	1.071	691	0.981
bcsstk36	23052	560044	620	1.045	483	1.099	706	0.993
struct3	53570	560062	1091	1.049	706	1.031	769	0.993
sf5	30169	190377	2781	1.055	1275	1.358	1374	1.001
nasasrb	54870	1311227	4771	1.724	2820	1.188	3548	0.993
cfd1	70656	878854	44520	1.278	13360	1.031	17930	1.000
cfd2	123440	1482229	136400	1.037	28710	1.130	34340	1.000

Stages

Matrix	vertices	edges	AMD	Post stages	BEND	Post stages	Metis	Post stages
shuttle-eddy	10429	46585	39	0.667	26	0.769	21	0.857
sf10	7294	44922	24	0.958	31	0.903	32	0.812
bcsstk35	30237	709963	32	0.844	29	0.966	30	0.833
bcsstk37	25503	557737	31	0.935	27	0.926	29	0.793
bcsstk36	23052	560044	24	0.958	24	1.000	26	0.769
struct3	53570	560062	29	0.966	30	0.967	27	0.815
sf5	30169	190377	29	0.966	38	0.947	41	0.878
nasasrb	54870	1311227	33	0.606	28	0.821	30	0.800
cfd1	70656	878854	48	0.812	35	1.000	38	0.737
cfd2	123440	1482229	38	1.000	40	0.925	34	0.853

Front sizes

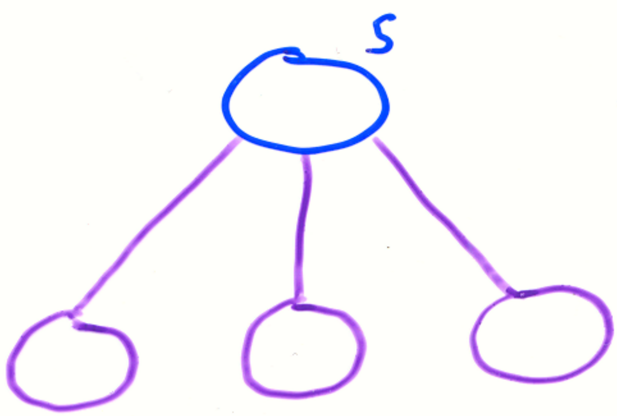
Matrix	vertices	edges	AMD	Post front	BEND	Post front	Metis	Post front
shuttle-eddy	10429	46585	121	1.603	143	1.203	168	1.000
sf10	7294	44922	448	1.042	296	1.000	325	1.000
bcsstk35	30237	709963	379	1.417	430	1.042	449	1.000
bcsstk37	25503	557737	527	1.309	518	1.266	624	1.000
bcsstk36	23052	560044	681	1.085	536	1.293	589	1.000
struct3	53570	560062	647	1.000	446	1.101	671	1.000
sf5	30169	190377	1278	1.000	829	1.449	876	1.000
nasasrb	54870	1311227	951	1.535	620	1.363	827	1.000
cfd1	70656	878854	2903	1.035	1417	1.008	2113	1.000
cfd2	123440	1482229	4065	1.000	1754	1.377	2433	1.000

Variation on Nested Dissection

Observation: once all but one component have been eliminated, don't need separator to prevent fill.

$\therefore$

Separator does not have to go last!



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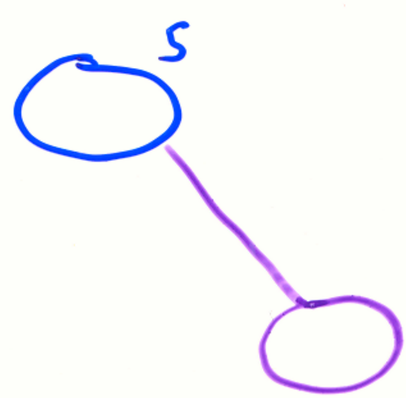
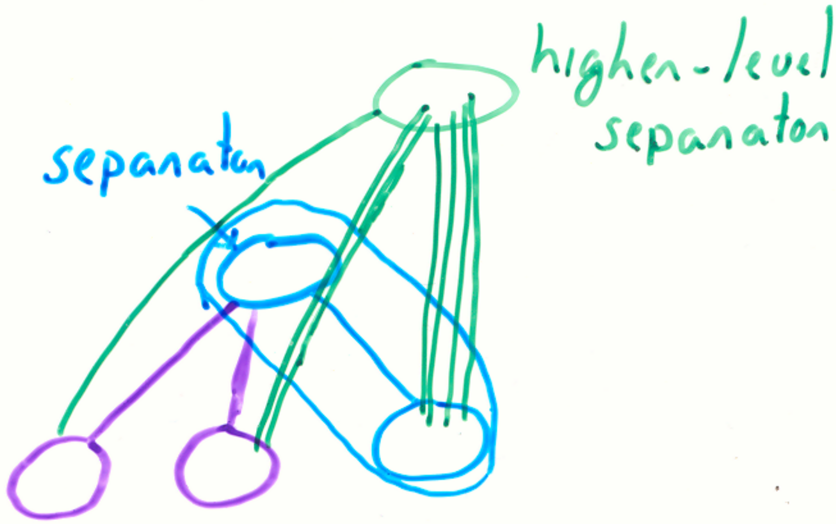


Heuristic:

If one component has more edges to higher level separators than any other component, order it last.

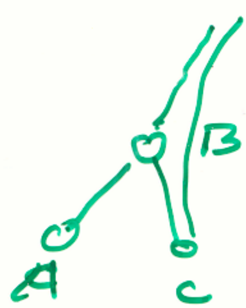


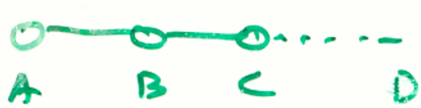
After other components have been eliminated, merge last component with separator and recurse.

[Otherwise do nested dissection]

Theoretical Justification

When applied to chordal graphs, this heuristic finds zero-fill orders!

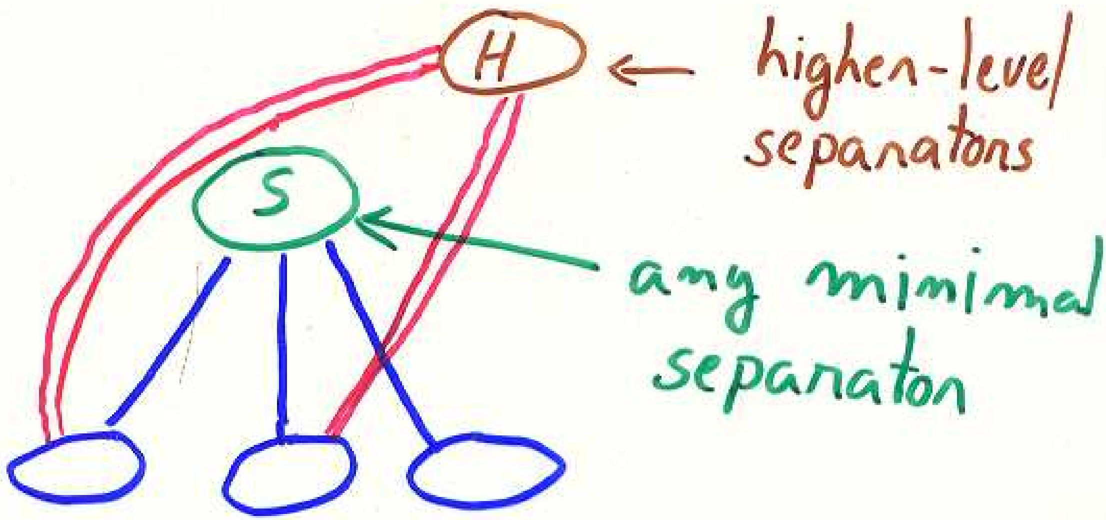




A Clique can Always go Last

Theorem: For any clique in a chordal graph there exists a zero-fill order in which the clique is eliminated last.

Intuition

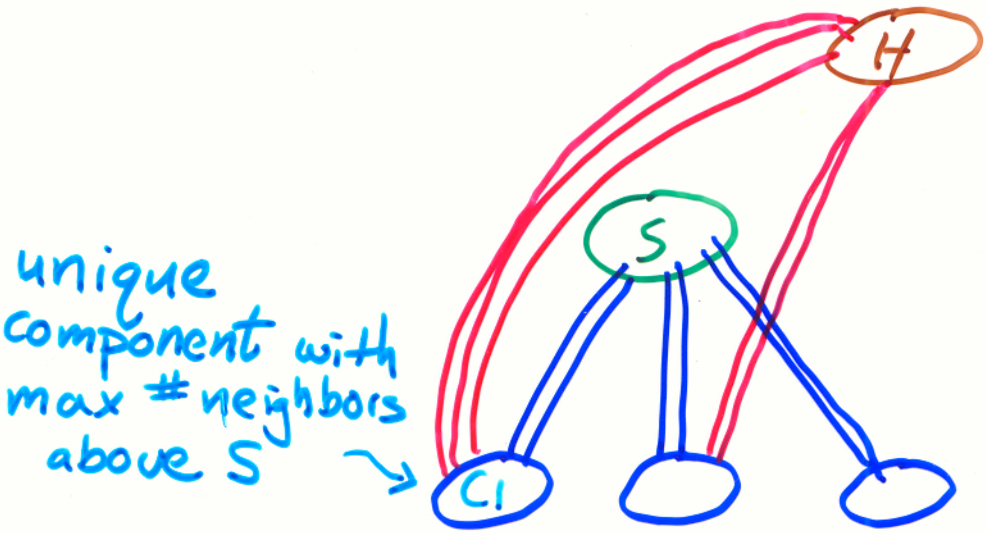


Fact: if two components have the maximum # of neighbors above S, then their neighbors in $S \cup H$ form a clique.

$\therefore$

$S \cup H$ can be eliminated after the components.

Intuition



Fact: Neighbors of C_1 in $S \cup H$ might not be a clique, but the neighbors of $C_1 \cup S$ in H are a clique.

Matrices

Matrix	vertices	edges	c-vertices	fill ($\times 10^3$)	work ($\times 10^6$)	height	stages	front
shuttle-eddy	10429	46585	10363	330	18	659	26	143
sf10	7294	44922	7294	531	71	703	31	296
bcsstk35	30237	709963	6611	2776	394	1084	29	430
bcsstk37	25503	557737	7093	2693	477	1162	27	518
bcsstk36	23052	560044	4351	2555	483	1255	24	536
struct3	53570	560062	41644	4452	706	1440	30	446
sf5	30169	190377	30169	3873	1275	1830	38	829
nasasrb	54870	1311227	24954	9765	2820	3360	28	620
cfd1	70656	878854	70656	22386	13360	3438	35	1417
cfd2	123440	1482229	123440	39026	28710	5538	40	1754

Statistics for hybrid algorithm BEND.

Fill

(relative to BEND)

Matrix	AMD	METIS	LRT	~~Our~~ LPND
shuttle-eddy	0.9910	1.1144	1.0139	0.9415
sf10	1.2726	1.0737	1.0205	0.9905
bcsstk35	0.9843	1.1284	0.9944	0.9733
bcsstk37	1.0395	1.1682	0.9564	0.9567
bcsstk36	1.0695	1.1887	1.0145	1.0028
struct3	1.1441	1.0274	0.9468	0.9667
sf5	1.3541	1.0320	0.9595	0.9521
nasasrb	1.2242	1.0837	1.0086	0.9560
cfd1	1.6856	1.0205	0.8597	0.8047
cfd2	1.9220	0.9977	0.9218	0.9411
AVERAGE	1.2687	1.0835	0.9696	0.9485
STDEV	0.3119	0.0654	0.0515	0.0543

Our implementation of standard nested dissection.

Our variation on nested dissection.

Height

Matrix	AMD	METIS	LRT	Our
shuttle-eddy	1.2914	0.5083	0.4431	1.1897
sf10	1.1280	0.7923	0.8990	0.9900
bcsstk35	1.1642	0.8948	0.8792	1.0563
bcsstk37	1.1472	1.0602	0.8614	1.0258
bcsstk36	1.2271	0.9291	0.9211	0.9793
struct3	1.0708	0.7590	0.6479	0.9868
sf5	1.2792	0.8322	0.8066	1.0443
nasasrb	1.4372	0.4946	0.4726	1.2711
cfd1	2.3040	0.9389	0.7833	1.5852
cfd2	1.7143	0.7346	0.6844	1.1685
AVERAGE	1.3763	0.7944	0.7399	1.1297
STDEV	0.3756	0.1819	0.1731	0.1882

Work

(relative to BEND)

Matrix	AMD	METIS	LRT	Our
shuttle-eddy	0.9642	1.2855	1.1027	0.8422
sf10	1.9313	1.1294	1.0865	0.9736
bcsstk35	0.9721	1.2905	1.0233	0.9642
bcsstk37	1.1150	1.4479	0.8705	0.8869
bcsstk36	1.2833	1.4609	1.0786	1.0075
struct3	1.5455	1.0895	0.9266	0.9841
sf5	2.1812	1.0776	0.8996	0.8824
nasasrb	1.6918	1.2582	1.1411	0.9199
cfd1	3.3323	1.3421	0.8293	0.6635
cfd2	4.7510	1.1961	0.9871	1.0355
AVERAGE	1.9768	1.2578	0.9945	0.9160
STDEV	1.2066	0.1364	0.1085	0.1076

Stages

Matrix	AMD	METIS	LRT	Our
shuttle-eddy	1.5000	0.8077	0.6538	1.4615
sf10	0.7742	1.0323	0.8387	1.1290
bcsstk35	1.1034	1.0345	0.9655	1.1379
bcsstk37	1.1481	1.0741	1.0000	1.1481
bcsstk36	1.0000	1.0833	0.8333	0.9583
struct3	0.9667	0.9000	0.7667	1.0333
sf5	0.7632	1.0789	1.0526	1.3158
nasasrb	1.1786	1.0714	0.6786	1.5714
cfd1	1.3714	1.0857	0.7143	1.4857
cfd2	0.9500	0.8500	0.7000	1.4750
AVERAGE	1.0756	1.0018	0.8204	1.2716
STDEV	0.2371	0.1069	0.1431	0.2169

Front sizes

Matrix	AMD	METIS	LRT	Our
shuttle-eddy	0.8462	1.1748	1.1049	0.7762
sf10	1.5135	1.0980	1.1858	0.9155
bcsstk35	0.8814	1.0442	1.0977	0.9953
bcsstk37	1.0174	1.2046	0.8301	0.9093
bcsstk36	1.2705	1.0989	1.2425	0.9757
struct3	1.4507	1.5045	1.0067	1.0942
sf5	1.5416	1.0567	0.8492	0.8480
nasasrb	1.5339	1.3339	1.3403	1.1016
cfd1	2.0487	1.4912	1.0974	0.8772
cfd2	2.3176	1.3871	1.2229	1.1431
AVERAGE	1.4422	1.2394	1.0978	0.9636
STDEV	0.4758	0.1767	0.1650	0.1205

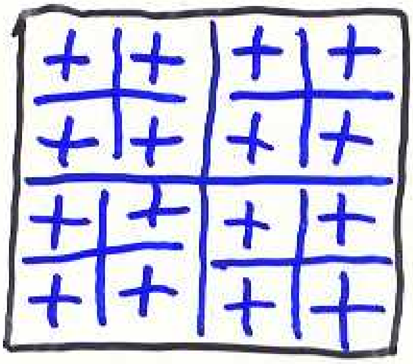
Time

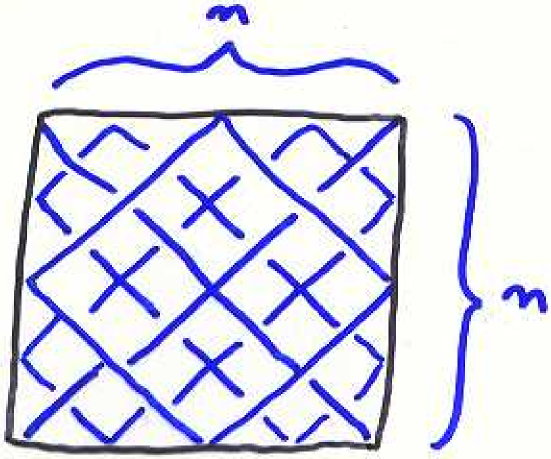
Matrix	BEND(secs)	OUR/BEND
shuttle-eddy	0.981	3.258
sf10	0.962	3.534
bcsstk35	7.365	0.351
bcsstk37	5.539	0.610
bcsstk36	5.400	0.313
struct3	7.131	3.565
sf5	3.491	5.916
nasasrb	11.817	1.805
cfd1	13.385	8.632
cfd2	18.237	13.141

Future Research

- Analyze min-degree on interval graphs.
- Analyze trade-off between work and fill.
- Tighten upper and lower bounds on fill and work on grid graphs.
- Methods for computing lower bounds on fill for specific graphs.

Nested Dissection on Grid Graphs





$$\text{fill} = 7.75\, n^2 \log n + O(n^2)$$

$$\text{fill} = \frac{7.75}{2}\, n^2 \log n + O(n^2)$$