Algorithms in the Real World (15-853), Fall 06
Assignment #4 **Due: 1 Nov 06**

Please answer all of the questions. You can look up material on the web and books, but you cannot look up solutions to the given problems. You can work in groups, but must write up the answers individually, and must write down your collaborators' names.

Problem 1: Running Times for Divide-and-Conquer Algorithms (10pt)
Consider applying divide-and-conquer to graphs and lets say that merging the two recursive solutions takes $f(s)$ time, where s is the number of edges separating the two graphs. For each of the following $f(s)$, and assuming you are given an edge separator tree for which all separators for the subgraphs of size n are 1/3-2/3 balanced and bounded by $kn^{1/2}$, what is the running time of such an approach.

1. s
2. $s \log s$
3. s^2
4. s^4

Problem 2: How much time for Independence (10pt)
Maximum independent set is the problem of finding the maximum number of vertices that can be marked in a graph such that no two neighbors are both marked. This problem is NP-complete. Consider a class of graphs that satisfies an k-separator theorem, where k is a fixed constant. Show how to solve maximum independent set in time that is exponential in k but polynomial in n.

Problem 3: Separators from Sparse Cuts (10 points)
Let $\mathcal{G}$ be a family of graphs. Suppose you are given a subroutine $\texttt{SparseCut}_{\mathcal{G}}$ that, given any graph $G = (V, E)$ in $\mathcal{G}$, outputs a partition of the vertex set into three parts L, R, S such that

- No vertex in L is adjacent to any vertex in R,
- $|L| \leq |R|$,
- each connected component of the graphs induced by the sets L and R belong to the class G, and
- the size of the set $|S| \leq |L|/n^{\epsilon}$.

Given a graph $G = (V, E)$ in $\mathcal{G}$, show how to use this subroutine $\texttt{SparseCut}_{\mathcal{G}}$ to find three sets A, B, C, with both $|A|, |B| \leq \frac{2}{3}|V|$, with $|C| \leq O(n^{1-\epsilon})$, and with no vertex in A being adjacent to any vertex of B. (Hint: In lecture, you saw a claim similar to this one to obtain an $(1/6, 5/6)$-separator with $O(\sqrt{n})$ nodes for the class of planar graphs. In this problem, you are being asked to formalize, strengthen, and extend the argument in class.)

Problem 4: Edge Separator Theorems Imply Small Degrees (10pt)
Prove that given a class of graphs satisfying an $O(n^{(d-1)/d})$, $d \geq 1$, edge-separator theorem, all members must have bounded degree (a maximum degree that is independent of the size n).

Problem 5: Shallow Cuts (10pt)
In class Anupam claimed that the algorithm CutShallow, given any connected planar graph with a "shallow" BFS tree of depth at most d, would 1/3-2/3 partition G using a separator of size $2d + 1$.
He claimed that this theorem can be extended to *unconnected* planar graphs in which each component has depth at most d. This is what was used to show that after deleting the smallest set C_i, the remaining graph can be 1/3-2/3 partitioned by removing $2\sqrt{n}+1$ vertices. (For this extension, you do not need to understand the inner workings of CutShallow.)