Lecture 3
Planar Kinematics

Matthew T. Mason

Mechanics of Manipulation

Outline

Decomposition of displacements

Planar kinematics

Displacements determined by two points

Displacements are rotations or translations

Rotation centers

Projective geometry

Motivation

The projective plane

Homogeneous coordinates

Projections

Where are we?

- Kinematics
  1. Foundations and general concepts.
  2. Planar kinematics.
  3. Spherical and spatial kinematics.

Readings etc.

- The text:
  - By now you should have read Chapter 1 of the text. The projective plane is covered in the Appendix of the text.
  - Today's material will take us through Sections 2.1, 2.2, and 2.5.
- Outside:
  - For an interesting history of kinematics, Chapter 1 of Hartenberg and Denavit's *Kinematic Synthesis of Linkages*.
  - Cool linkages etc etc: Reuleaux's *Kinematics of Machinery*.
  - Both the above are openly available at the KMODDL web site `http://kmoddl.library.cornell.edu`.
  - Geometric constructions and linkages simulated on the course web page.
  - Hilbert and Cohn-Vossen. *Geometry and the Imagination*.

Decomposition of displacements

Translation ∘ Rotation

Theorem (2.2)

For any displacement D of $\mathbb{E}^n$, and any point O, D is the composition of a translation with a rotation about O.

Proof.

- Let O' be the image of O under D.
- Let T be the translation taking O to O'.
- Consider the displacement $T^{-1} \circ D$. Where does it map O?

$$(T^{-1} \circ D)(O) = ???$$

- So $T^{-1} \circ D$ is a rotation; call it R.
- So then $T \circ R = T \circ T^{-1} \circ D = D$.

□

Decomposition of displacements

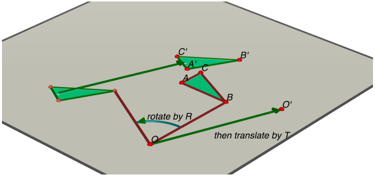
Note:

- Theorem 2.2 is basis for most common representation of displacements.
- The decomposition is not unique: it depends on the choice of O.
- Note how simple it is to prove using group theory. We are dividing the displacement D by the translation T!
- Applies to $\mathbb{E}^n$ for *all* $n \in \mathbb{Z}$.
- Instead of $D = T \circ R$, (rotation, then translation) we could have $D = R \circ U$ (translation, then rotation).
- The order matters. (Planar displacements do not commute!) So $T \neq U$.
- Book has misleading remark in proof of this theorem. See errata file.

Decomposition examples

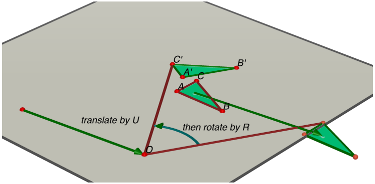
Rotate then translate

$$D = T \circ R$$



Translate then rotate

$$D = R \circ U$$



Planar kinematics

Motivation

That is all we will do on “general” kinematics. On to planar kinematics.

Why planar kinematics?

- The kinematics of flatland? No, much bigger.
- Planar motions are common in $\mathbb{E}^3$. All points moving in parallel planes.
  - Most mobile robots on flat terrain (except when they fall over).
  - Many grippers use planar motion.
  - Many kinematic linkages use planar motion.
- All spatial motions can be decomposed into components including planar motions.
- Spatial rotation is closely related to planar motion.

Two points is enough.

What can we say about rigid motions of $\mathbb{E}^2$?
The first thing is: two points is enough . . .

Theorem (2.3)

A planar displacement is completely determined by the motion of any two points.

Proof.

Construct a coordinate frame . . . □

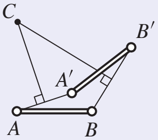
Every D is an R or a T

Theorem (2.4)

Every planar displacement is either a translation or a rotation.

Useful construction

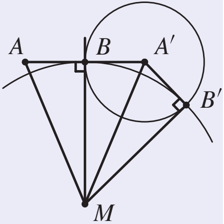
- Pick two points A and B.
- Let A' and B' be the images.
- Construct perpendicular bisectors.
- Intersection gives fixed point. Why? Preserves distance from A and from B.



- Looks like a constructive proof, but it is not.

A legitimate proof

Proof.

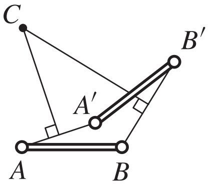


- Pick any point A. We can assume $A \neq A'$.
- Pick B the midpoint of line segment $\overline{AA'}$. We can assume B' is not on $\overline{AA'}$.
- Construct $\perp$ to AB at B, and $\perp$ to $A'B'$ at B'. They are not parallel. Let M be their intersection.
- Consider the rotation R that maps A to A' and M to itself. Where is $R(B)$? Distance constraints give two circles, with two intersections: B or B'.
- So R maps B to B'. $R = D$.

□

Rotation centers

- Consider again construction of rotation centers from the motion of two points. How does it fail when $\overline{AA'}$ is parallel to $\overline{BB'}$?



  - The perpendiculars are parallel. There is no intersection, hence no rotation center.
  - But, in the projective plane they *do* intersect!!!
  - *Every planar displacement is a rotation about a point in the projective plane.*
  - But that is *not* a displacement or a rotation *of* the projective plane. There is no distance, or angle, in the projective plane, hence no rigidity.

Projective geometry

Motivation

Why projective geometry?

- It's cool.
- It's useful.
- Specifically, it gives us points at *infinity* which are very useful in kinematics.
- It also gives us a useful *dual* mapping between points and lines.
- It's really cool.

The projective plane
Overview

The basic idea:

- Start with the Euclidean plane $\mathbb{E}^2$.
- In the Euclidean plane, when two lines intersect you get a point, but some pairs of lines don't intersect: *parallel lines*. Euclid's fifth postulate.
- Add some points, the **ideal points** or the **points at infinity**. One point for each set of parallel lines.
- Call the new structure the **projective plane**—$\mathbb{P}^2$.

We will do it concretely using **homogeneous coordinates**.

Homogeneous coordinates

Definition

Let the Cartesian coordinates of some point in $\mathbb{E}^2$ be

$$(\eta, \nu) \in \mathbb{R}^2$$

Then we will say that

$$(x, y, w) \triangleq (w\eta, w\nu, w)$$

are the **homogeneous coordinates** of the point, **provided**

$$w \neq 0$$

To go from homogeneous coordinates in $\mathbb{R}^3$ to Cartesian coordinates in $\mathbb{R}^2$:

$$\begin{pmatrix} x \\ y \\ w \end{pmatrix} \mapsto \begin{pmatrix} x/w \\ y/w \end{pmatrix}, w \neq 0 \qquad (1)$$

Homogeneous coordinates

Invariance to scaling

Scaling the homogeneous coordinates does **not** change the point!

$$\begin{pmatrix} ax \\ ay \\ aw \end{pmatrix} \mapsto \begin{pmatrix} ax/aw \\ ay/aw \end{pmatrix} = \begin{pmatrix} x/w \\ y/w \end{pmatrix}, a, w \neq 0$$

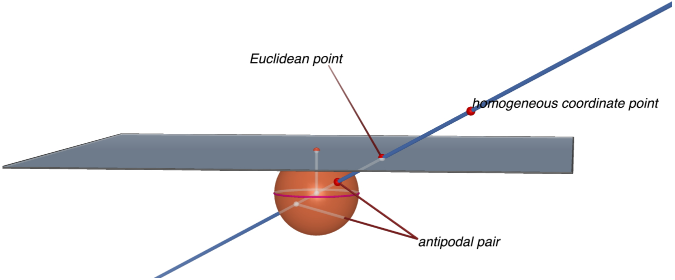
So, homogeneous coordinates represent a point in $\mathbb{R}^2$ by a line through the origin of $\mathbb{R}^3$.

$$\begin{pmatrix} x \\ y \end{pmatrix} \leftrightarrow \left\{ \begin{pmatrix} wx \\ wy \\ w \end{pmatrix} \middle| w \neq 0 \right\}$$

Homogeneous coordinates

Central projection

- In homogeneous coordinate space, embed the Euclidean plane as $w = 1$.
- Also embed the sphere $x^2 + y^2 + w^2 = 1$.
- A line through the origin of $\mathbb{R}^3$ probably (!)
  - intersects the sphere in **antipodal points**
  - intersects the $w = 1$ plane at the appropriate point $(x/w, y/w)$.



Homogeneous coordinates
Ideal points

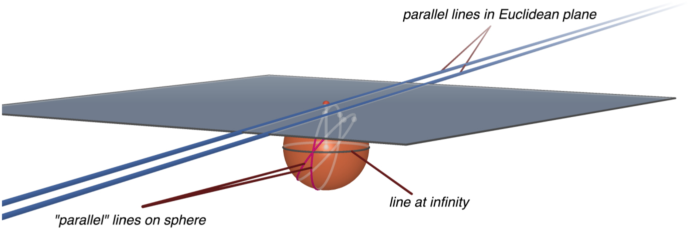
- The original idea: extend $\mathbb{E}^2$ by adding some ideal points.
- Euclidean point:
  line through origin of $\mathbb{R}^3$ intersecting $w = 1$ plane.
- Ideal point:
  line through origin of $\mathbb{R}^3$ *not* intersecting $w = 1$ plane.

With Cartesian coords, no place to put ideal points. With homogeneous coordinates, there's a perfect spot!

Projective plane

Definition

- Define the projective plane $\mathbb{P}^2$ to be the set of lines through the origin of $\mathbb{E}^3$.
- A line in $\mathbb{E}^2$ is represented by plane through origin of $\mathbb{E}^3$.
- The ideal points form a line! The **line at infinity**. The equator of the embedded sphere.
- “Parallel lines” intersect at infinity.



Projective plane and duality

- Duality. Two points determine a line. Two lines determine a point. Every axiom of the projective plane has a dual axiom by switching “line” and “point”.
- Every theorem likewise has a dual theorem, and every proof a dual proof. In projective geometry, you get to prove two theorems for the price of one proof.

Projective plane and duality
Dual map

- Using the homogeneous coordinate construction, the dual mapping between points and lines is concrete. A point in $\mathbb{P}^2$ is represented by a line through the origin of $\mathbb{R}^3$, perpendicular to a plane through the origin of $\mathbb{R}^3$, representing a line in $\mathbb{R}^2$.