Lecture 2
Kinematic Foundations

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Mechanics of Manipulation

Outline

Preview

The agenda

- Today: general concepts.
- Next: rigid motions in the Euclidean plane ($\mathbb{E}^2$).
- Then: rigid motions in Euclidean three space ($\mathbb{E}^3$), and the sphere ($\mathbb{S}^2$).

Why?

- Why spend so much time on such fundamental concepts? Because robotics is so challenging.

  > *"Give me six hours to chop down a tree and I will spend the first four sharpening the axe." (Abraham Lincoln)*

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Ambient space, system, configuration

Definitions

Definition (Ambient space)

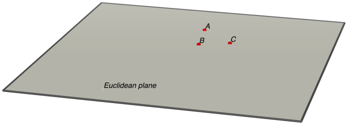
Let $\mathbb{X}$ be the ambient space, either $\mathbb{E}^2$, $\mathbb{E}^3$, or $\mathbb{S}^2$.

Definition (System)

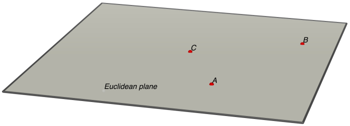
A set of points in the space $\mathbb{X}$.

Definition (Configuration)

The configuration of a system gives the location of every point in the system.



A system in the Euclidean plane



Same system, different configuration

Rigid bodies, displacements

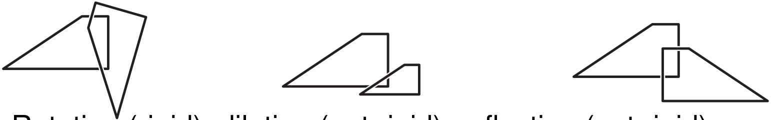
Definitions

Definition (Displacement)

A displacement is a change of configuration that preserves pairwise distance and orientation (handedness) of a system.

Definition (Rigid body)

A rigid body is a system that is capable of displacements only.



Rotation (rigid), dilation (not rigid), reflection (not rigid)

Question

Why focus on rigid bodies?

- Nothing is rigid.
- Lots of stuff is articulated or *way* soft: tissue (surgery), fluids, food, paper, books, ...

Lots of reasons

- A reasonable approximation for some objects.
- Even non-rigid transformations can be factored into displacement + shape change.
- Some things are invariant with respect to displacements.
- Other?

Moving and fixed spaces

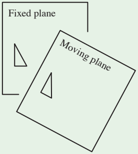
Basic convention

Convention

We will consider displacements to apply to *every* point in the ambient space.

Example

For example, planar displacements are described as motion of *moving* plane relative to *fixed* plane.



Moving and fixed planes.

Rotations and translations

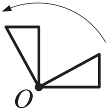
Definitions

Definition (Rotation)

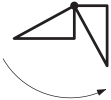
A rotation is a displacement with at least one fixed point.

Definition (Translation)

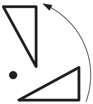
A translation is a displacement for which all points move equal distances along parallel lines.



Rotation about O



Rotation about a point on the body



Rotation about a point not on the body

Configuration space, degrees of freedom

Definitions

Definition (Configuration space)

Configuration space is the space comprising all configurations of a given system.

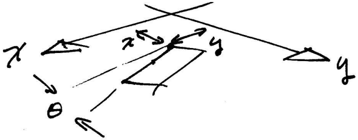
Definition (Degrees of freedom (DoFs))

The Degrees of Freedom (DOFs) of a system is the dimension of the configuration space. (Less precisely: the number of reals required to specify a configuration.)

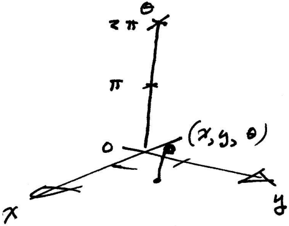
Systems, configuration spaces, DOFs

Examples

System	Configuration	DOFs
point in plane	x, y	2
point in space	x, y, z	3
rigid body in plane	x, y, θ	3
rigid body in space	$x, y, z, \phi, \theta, \psi$	6
rigid body in 4-space	???	???



Ambient space



Configuration space

Solution: DOFs of a rigid body in $\mathbb{E}^4$.

- Most common answer: 4 translations and 4 rotations.
- Why assume four rotational freedoms? Generalizing from $\mathbb{E}^3$? Consider:

Space	**Translational DOFs**	**Rotational DOFs**
$\mathbb{E}^0$ (point)	0	0
$\mathbb{E}^1$(*line*)	1	0
$\mathbb{E}^2$(*plane*)	2	1
$\mathbb{E}^3$	3	3
$\mathbb{E}^4$	4	6

- The correct generalization for $\mathbb{E}^n$ is n choose 2. Identify rotational freedoms not with a single axis, which works only in $\mathbb{E}^3$, but with a pair of axes.
- The proof is simple, after we have covered rotation matrices.
- Rotations in $\mathbb{E}^4$ are useful. See the lecture on quaternions.

Configuration spaces

Metrics

- We can define a *metric*, or a *distance function*, for any configuration space. What is a metric?

Definition

A metric d on a space X is a function $d : X \times X \mapsto \mathbf{R}$ satisfying:

- $d(x, y) \geq 0$ (non-negativity);
- $d(x, y) = 0$ if and only if $x = y$;
- $d(x, y) = d(y, x)$ (symmetry);
- $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality).

- Every space has a metric! But what would make a *suitable* metric for configuration space? How would you devise a suitable metric for configurations of $\mathbb{S}^2$? $\mathbb{E}^2$? $\mathbb{E}^3$?

Solution: metrics for Cspaces

- Possibilities:
  1. Let every pair of configurations have distance 1.
  2. For $\mathbb{S}^2$, the minimum angle to rotate one configuration to the other.
  3. For $\mathbb{E}^n$ decompose displacement into rotation and translation, and add the distance to the angle.
  4. Same idea, but scale angle by a characteristic length.
  5. In general: pick some finite set of points, and take the maximum distance travelled by the points.
- (1): useless. (2–5): if two configurations are close, in the sense that corresponding points are close in the ambient space, then distance is small. (3): dimensionally inconsistent.
- A desirable property: invariance with respect to displacements. (2) achieves this for $\mathbb{S}^2$. Unattainable for $\mathbb{E}^2$ and $\mathbb{E}^3$.

Digressing for group theory

Motivation

- Why? If you can show that your mathematical construct is a *group*, then you can use algebra—you can write and solve equations.
- Displacements are a group, and we need to use algebra on them.

Group
Definition

A *group* is a set of elements X and a binary operator $\circ$ satisfying the following properties:

- **Closure:**
  for all x and y in X, $x \circ y$ is in X.
- **Associativity:**
  for all x, y, and z in X, $(x \circ y) \circ z$ is equal to $x \circ (y \circ z)$.
- **Identity:**
  there is some element, called 1, such that for all x in X $x \circ 1 = 1 \circ x = x$.
- **Inverses:**
  for all x in X, there is some element called x^{-1} such that $x \circ x^{-1} = x^{-1} \circ x = 1$.

Commutativity of groups

- Note that *commutativity* is *not* required for a group. Some are, some are not.
- A commutative group is also known as an *Abelian* group.

Examples of groups

Example

- Let the elements be the integers Z;
- let the operator be ordinary addition $+$.
- Is it a group? Verify the properties.
- Is it commutative?

Example

- Let the elements be the reals **R**;
- Let the operator be multiplication $\times$.
- Is it a group?

- Other examples: positive rationals with multiplication (commutative); nonsingular k by k real matrices with matrix multiplication (noncommutative).

Displacements with composition are a group

- Every displacement D is an operator on the ambient space $\mathbb{X}$, mapping every point x to some new point $D(x) = x'$.
- The product of two displacements is the composition of the corresponding operators, i.e. $(D_2 \circ D_1)(\cdot) = D_2(D_1(\cdot))$.
- The inverse of a displacement is just the operator that maps every point back to its original position.
- The identity is the null displacement, which maps every point to itself.

In other words:

The displacements, with functional composition, form a group.

SE(2), **SE**(3), and **SO**(3)

Special Euclidean and Special Orthogonal groups

These groups of displacements have names:

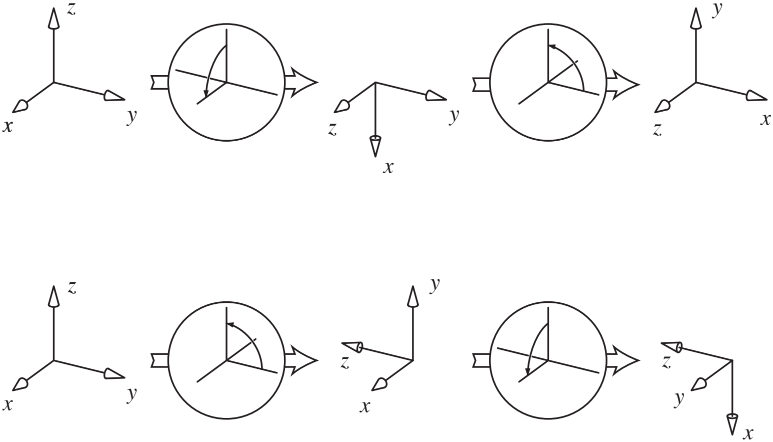
- **SE**(2): The Special Euclidean group on the plane.
- **SE**(3): The Special Euclidean group on $\mathbb{E}^3$.
- **SO**(3): The Special Orthogonal group.

Whence the names?

- **Special**: they preserve orientation / handedness.
- **Orthogonal**: referring to the connection with orthogonal matrices, which will be covered later.

Noncommutativity of rotations.

Does **SO**(3) commute? **NO!** No, no, no.



Do displacements commute?

Does **SE**(3) commute?
Does **SE**(2) commute?
Does **SO**(2) commute?